On the Spectral Analysis of Direct Sums of Riemann-Liouville Operators in Sobolev Spaces of Vector Functions

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Abstract

Let J_k^α be a real power of the integration operator J_k defined on Sobolev space $W_p^k[0,1]$. We investigate the spectral properties of the operator $A_k = \bigoplus_{j=1}^n \lambda_j J_k^\alpha$ defined on $\bigoplus_{j=1}^n W_p^k[0,1]$. Namely, we describe the commutant $\{A_k\}'$, the double commutant $\{A_k\}''$ and the algebra $\mathsf{Alg}\, A_k$. Moreover, we describe the lattices $\mathsf{Lat}\, A_k$ and $\mathsf{HypLat}\, A_k$ of invariant and hyperinvariant subspaces of A_k, respectively. We also calculate the spectral multiplicity μ_{A_k} of A_k and describe the set $\mathsf{Cyc}\, A_k$ of its cyclic subspaces. In passing, we present a simple counterexample for the implication

$$\mathsf{HypLat}(A \oplus B) = \mathsf{HypLat}\, A \oplus \mathsf{HypLat}\, B \Rightarrow \mathsf{Lat}(A \oplus B) = \mathsf{Lat}\, A \oplus \mathsf{Lat}\, B$$

to be valid.

1 Introduction

It is well known [9, 20, 33, 36] that the Volterra integration operator $J: \ f(x) \to \int_0^x f(t)\,dt$ as well as its real powers J^α play an exceptional role in the spectral theory of nonselfadjoint operators in $L_2[0,1]$. The paper is devoted to the spectral analysis of direct sums of multiples of powers J^α of the integration operator J in Sobolev spaces. To describe its content we first briefly recall basic facts on the integration operator.

It is well known [9, 20, 33, 36] that J is unicellular on $L_p[0,1]$ for $p \in [1,\infty)$ and the lattice $\mathsf{Lat}\, J$ of its invariant subspaces is anti-isomorphic to the segment $[0,1]$. The same is also true (see [20, 36]) for the simplest Volterra operators

$$J^\alpha: \ f(x) \to \int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)\,dt, \qquad \alpha > 0,$$

being the positive powers of the integration operator J.

More precisely, it is known (see [9, 20, 33, 36]) that

$$\begin{aligned} &\mathsf{Lat}\, J^\alpha = \mathsf{HypLat}\, J^\alpha = \{E_a: \ a \in [0,1]\}, \\ &E_a := \{f \in L_p[0,1]: \ f(x) = 0 \ \text{ for a.a. } \ x \in [0,a]\}. \end{aligned} \tag{1.1}$$

Description (1.1) yields (and, in fact, is equivalent to) [9, 20, 36] the following description of cyclic vectors of J^α

$$f \text{ is a cyclic vector for } J^\alpha \Leftrightarrow \int_0^\varepsilon |f(x)|^p\,dx > 0 \qquad \text{for all } \ \varepsilon > 0. \tag{1.2}$$

This condition is called the ε - condition.

Description (1.1) of $\mathsf{HypLat}\,J^\alpha$ is closely connected with the description of the commutant $\{J^\alpha\}'$. The commutant $\{J\}'$ of the operator J defined on $L_2[0,1]$ as well as the (weakly closed) algebra $\mathsf{Alg}\,J$ generated by J and $\mathbb{I}$ were originally described by D. Sarason [44] (see also a simple proof in [18]). Another, description of $\mathsf{Alg}\,J$ for J acting in $L_p[0,1]$ has also been obtained in [29, 30]. Namely, it was shown in [29, 30] that if J is defined on $L_p[0,1]$ $(1 < p < \infty)$, then $\{J^\alpha\}' = \mathsf{Alg}\,J^\alpha$ and $K \in \{J^\alpha\}'$ if and only if it is bounded and admits a representation

$$(Kf)(x) = \frac{d}{dx}\int_0^x k(x-t)f(t)\,dt, \qquad k \in L_{p'}[0,1], \tag{1.3}$$

where ${p'}^{-1} + p^{-1} = 1$. Using a criterion of boundedness of K defined on $L_2[0,1]$ (see [30, Proposition 3.1']) it can easily be shown that for $p = 2$ description (1.3) is equivalent to that obtained in [44].

Now, let $A = J^\alpha \otimes B (= \bigoplus_{j=1}^n \lambda_j J^\alpha)$ be a tensor product of the operator J^α defined on $L_p[0,1]$ and the $n \times n$ nonsingular diagonal matrix $B = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^{n\times n}$. The investigation of such operators with $B = B^*$ was initiated by G. Kalisch [24]. He has extended the known Livsic theorem (see [9, 20]) to the case of (abstract) Volterra operators with finite-dimensional real part and characterized those of them that are unitarily equivalent to A with $B = B^*$ and $\alpha = 1$ (see also [9, 20]).

Later on, sufficient conditions for a Volterra operator $K : f \to \int_0^x K(x,t)f(t)\,dt$ defined on $L_p[0,1] \otimes \mathbb{C}^n$ to be similar to the operator A have been indicated in [32]. So, A may be treated as a similarity model for a wide class of Volterra operators. This result has been applied in [32] to the problem of unique recovery of a Dirac type system by its monodromy matrix (see also references therein).

Further, one of the authors [29, 31] described the lattices $\mathsf{Lat}\,A$ and $\mathsf{HypLat}\,A$ and the set $\mathsf{Cyc}\,A$ of cyclic subspaces of the operator $A = J^\alpha \otimes B (= \bigoplus_{j=1}^n \lambda_j J^\alpha)$ defined on $L_p[0,1] \otimes \mathbb{C}^n$, $p \in (1, \infty)$. In particular, in [29, 31] necessary and sufficient conditions for a sequence $\{\lambda_i\}_{i=1}^n$ guaranteeing the splitting of each of the lattices $\mathsf{Lat}\,A$ and $\mathsf{HypLat}\,A$, as well as of the commutant $\{A\}'$ and double commutant $\{A\}''$ of A were found. More precisely, it was proved in [29, 31] that each of the following relations

$$\mathsf{Lat} \bigoplus_{j=1}^n \lambda_j J^\alpha = \bigoplus_{j=1}^n \mathsf{Lat}\,\lambda_j J^\alpha, \tag{1.4}$$

$$\mathsf{HypLat} \bigoplus_{j=1}^n \lambda_j J^\alpha = \bigoplus_{j=1}^n \mathsf{HypLat}\,\lambda_j J^\alpha, \tag{1.5}$$

$$\left\{\bigoplus_{j=1}^n \lambda_j J^\alpha\right\}' = \bigoplus_{j=1}^n \{\lambda_j J^\alpha\}' = \left\{\bigoplus_{j=1}^n \lambda_j J^\alpha\right\}'' = \bigoplus_{j=1}^n \{\lambda_j J^\alpha\}'' \tag{1.6}$$

is equivalent to the condition

$$\arg \lambda_i \neq \arg \lambda_j \pmod{2\pi} \qquad 1 \leqslant i < j \leqslant n. \tag{1.7}$$

Some partial cases of the equivalence (1.4)⇔(1.7) have been obtained earlier in [23, 39, 40] (see Remark 2.20).

It is easily seen that (1.6) is equivalent to the following fact : for any $\lambda \notin (0, +\infty)$ an operator equation

$$J^\alpha X = \lambda X J^\alpha \tag{1.8}$$

has only zero bounded solution X. Moreover, in [29, 31] a description of all nonzero solutions X of (1.8) with $\lambda \in (0, +\infty)$ was obtained. Recently, equation (1.8), and even more general ones with a bounded A in place of J^α, has attracted attention of several mathematicians (see, for instance, [5, 6, 26], and [8, 10, 45]). In particular, some results from [29] on equation (1.8) were rediscovered in [5] and [26] (the case $\alpha = 1$) and in [6] (the case $\alpha \in \mathbb{Z}_+ \setminus \{0\}$). These authors treat any solution X of $AX = \lambda XA$ as an extended eigenvector of A (see Remark 2.22 (2)).

Note also that if (1.7) is not fulfilled then A is not cyclic. The set $\mathsf{Cyc}\, A$ of cyclic subspaces of A was described in [29, 31] by using a notion of $*$-determinant (see Definition 2.15). For example, vectors $f_1 := (f_{11}, f_{12})$, $f_2 := (f_{21}, f_{22})$ generate a cyclic subspace of the operator $A = J \oplus J$ defined on $L_p[0,1] \oplus L_p[0,1]$ if and only if the function $*\text{-det} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} := f_{11} * f_{22} - f_{12} * f_{21}$ satisfies ε - condition (1.2) (here $f * g$ stands for the convolution of functions $f,\ g \in L_1[0,1]\ :\ (f * g)(x) := \int_0^x f(x-t) g(t)\, dt$).

Passing to the case of the Sobolev space we should mention the pioneering work of E. Tsekanovskii [46]. More precisely, it is shown in [46] (see also [41]) that the integration operator $J_k\ :\ f(x) \to \int_0^x f(t)\, dt$ defined on $W_p^k[0,1]$ is unicellular too and $\mathsf{Lat}\, J_k$ consists of continuous part $\mathsf{Lat}^c\, J_k$ and discrete part $\mathsf{Lat}^d\, J_k$, $\mathsf{Lat}\, J_k = \mathsf{Lat}^c\, J_k \cup \mathsf{Lat}^d\, J_k$. Here

$$\begin{gathered} \mathsf{Lat}^c\, J_k = \big\{ E^k_{a,0} :\ a \in (0,1] \big\} \cup E_{0,0}, \\ E^k_{a,0} := \big\{ f \in W_p^k[0,1] : f(x) = 0 \ \text{ for } x \in [0,a] \big\}, \qquad E_{0,0} := W^k_{p,0}[0,1], \end{gathered} \tag{1.9}$$

is a continuous chain and $\mathsf{Lat}^d\, J_k = \{E_l^k\}_{l=0}^k$ with $E_k^k := W_p^k[0,1]$ and

$$E_l^k = \big\{ f \in W_p^k[0,1] :\ f(0) = \cdots = f^{(k-l-1)}(0) = 0 \big\}, \quad l \in \{1, \ldots, k-1\}, \tag{1.10}$$

is a discrete chain. It is clear that, for $0 \leqslant a_1 \leqslant a_2 \leqslant 1$,

$$\begin{aligned} \{0\} &= E^k_{1,0} \subset E^k_{a_2,0} \subset E^k_{a_1,0} \subset E^k_{0,0} \\ &= W^k_{p,0}[0,1] = E^k_0 \subset E^k_1 \subset \cdots \subset E^k_k = W^k_p[0,1]. \end{aligned}$$

In [16] we investigated the spectral properties of the complex powers J_k^α of the integration operator J_k defined on Sobolev space $W_p^k[0,1]$. Namely, in [16] were described the lattices $\mathsf{Lat}\, J_k^\alpha$ and $\mathsf{HypLat}\, J_k^\alpha$, the set of cyclic subspaces $\mathsf{Cyc}\, J_k^\alpha$, the operator algebra $\mathsf{Alg}\, J_k^\alpha$, the commutant $\{J_k^\alpha\}'$ and the double commutant $\{J_k^\alpha\}''$. In

particular, it turns out that $\{J_k^\alpha\}' = \{J_k^\alpha\}''$ and $\{J_k^\alpha\}'$ and $\mathsf{Alg}\, J_k^\alpha$ can be described as follows:

$$R \in \{J_k^\alpha\}' \Leftrightarrow (Rf)(x) = cf(x) + \int_0^x r(x-t)f(t)\,dt, \quad r \in W_p^{k-1}[0,1], \tag{1.11}$$

$$\begin{aligned} &R \in \mathsf{Alg}\, J_k^\alpha \\ &\quad \Leftrightarrow \begin{cases} R \in \{J_k^\alpha\}',\ r^{(l)}(0) = 0, \quad l \neq m\alpha - 1,\ m \leqslant [\frac{k-1}{\alpha}], & 1 \leqslant \alpha \leqslant k-1, \\ R \in \{J_k^\alpha\}',\ r \in W_{p,0}^{k-1}[0,1], & 2 \leqslant k \leqslant \alpha + \frac{1}{p}. \end{cases} \end{aligned} \tag{1.12}$$

It was also shown in [16] that the operator J_k^α is unicellular on $W_p^k[0,1]$ if and only if either $k = 1$ or $\alpha = 1$. Moreover, the unicellularity of J_k^α is equivalent to the validity of the "Neumann-Sarason" identity $\mathsf{Alg}\, J_k^\alpha = \{J_k^\alpha\}''$.

In this paper we extend the main results from [16] and [31] to the case of the operator $A_k := J_k^\alpha \otimes B$ defined on Sobolev space $W_p^k[0,1] \otimes \mathbb{C}^n$ of vector-functions. Moreover, we investigate the spectral properties of the operator $A_k := \bigoplus_{j=1}^n \lambda_j J_{k_j}^\alpha$.

The paper is organized as follows. In Section 2, we collect some auxiliary results about invariant subspaces for C_0 contractions and accretive operators. Here we also present and complete some results from [31] for the operator $A = \bigoplus_{j=1}^n \lambda_j J^\alpha$ defined on $\bigoplus_{j=1}^n L_p[0,1]$.

In Section 3, it is shown that the operator $A = \bigoplus_{j=1}^n \lambda_j J^\alpha$ defined on $\bigoplus_{j=1}^n L_p[0,1]$ and the operator $A_{k,0} = \bigoplus_{j=1}^n \lambda_j J_{k,0}^\alpha$ defined on $\bigoplus_{j=1}^n W_{p,0}^k[0,1]$ are isometrically equivalent. Hence all results on the operator A presented in Section 2 are immediately extended to the case of the operator $A_{k,0}$.

In Section 4, we provide a spectral analysis of the operator $A_k = \bigoplus_{j=1}^n \lambda_j J_{k,0}^\alpha$ defined on $\bigoplus_{j=1}^n W_p^k[0,1]$. A descriptions of the (weakly closed) algebra $\mathsf{Alg}\, A_k$, commutant $\{A_k\}'$ and double commutant $\{A_k\}''$ is presented in Subsection 4.1, Subsection 4.2 and Subsection 4.3, respectively.

In Subsection 4.4, we obtain a description of the lattice $\mathsf{Lat}\, A_k$ assuming that $A_k := \bigoplus_{j=1}^n \lambda_j J_{k_j}^\alpha$ satisfies condition (1.7). This description is essentially based on a description of $\mathsf{Lat}\, T$ (Theorem 2.1) for finite-dimensional operator T in $\bigoplus_{j=1}^n \mathbb{C}^{k_j}$. In Subsection 4.5, a description of the lattice $\mathsf{HypLat}\, A_k$ is contained. We emphasize that $\mathsf{HypLat}\, A_{k,0} = \mathsf{HypLat}^c\, A_k$ and the "continuous part" of $\mathsf{HypLat}\, A_k$ does not depend on α.

It turns out that under condition (1.7) $\mathsf{HypLat}\, A_k$ as well as the commutant $\{A_k\}'$ of the operator A_k splits, that is, relations (1.5)-(1.6) remain valid with $\mathsf{HypLat}\, A$ and $\{A\}'$ replaced by $\mathsf{HypLat}\, A_k$ and $\{A_k\}'$, respectively. On the other hand, under condition (1.7) $\mathsf{Lat}\, A_k$ does not split for $k \geqslant 1$ in contrast to (1.4).

In this connection we recall (see [11]) that for a direct sum $T_1 \oplus T_2$ of two operators on a Banach space the relations (1.5)-(1.6) are equivalent to each other and both are implied by (1.4). Thus, the operator A_k presents a simple counterexample to the validity of the implication

$$\mathsf{HypLat}(T_1 \oplus T_2) = \mathsf{HypLat}\, T_1 \oplus \mathsf{HypLat}\, T_2 \quad \Longrightarrow \mathsf{Lat}(T_1 \oplus T_2) = \mathsf{Lat}\, T_1 \oplus \mathsf{Lat}\, T_2.$$

Other counterexamples can be found in [11].

In Subsection 4.6, we compute the spectral multiplicity and present a description of the cyclic subspaces $\mathsf{Cyc}\, A_k$ for the operator A_k.

It should be emphasized that descriptions of the sets $\mathsf{Cyc}\, A_k$ and $\mathsf{Cyc}\, A_{k,0}$ essentially differ. Namely, the first description does not depend on a choice of a sequence $\{\lambda_j\}_1^n$, though the second one depends on $\{\arg\lambda_j\}_1^n$ and is similar to that obtained in [31] for $\bigoplus_{j=1}^n L_p[0,1]$.

A description of the set of cyclic subspaces of the operator $A = \bigoplus_{j=1}^m \lambda_j J_{k_j}^\alpha \oplus \bigoplus_{j=m+1}^n \lambda_j J_{k_j,0}^\alpha$ acting in the mixed space $\bigoplus_{j=1}^m W_p^{k_j}[0,1] \oplus \bigoplus_{j=m+1}^n W_{p,0}^{k_j}[0,1]$ is presented too.

Main results of the paper have been announced (without proofs) in [15].

1.1 Notations and agreements

1. X, X_1, X_2 stand for Banach spaces;
2. $[X_1, X_2]$ is the space of bounded linear operators from X_1 to X_2; $[X] := [X, X]$;
3. $\mathbb{I}$ and $\mathbb{I}_k$ denote the identity operators on X and on $\mathbb{C}^k$, respectively; $\mathbb{O} := 0 \cdot \mathbb{I}$, $\mathbb{O}_k := 0 \cdot \mathbb{I}_k$;
4. $J(0;k)$ denotes the Jordan nilpotent cell of order k;
5. $\ker T = \{x \in X \ : \ Tx = 0\}$ is the kernel of $T \in [X]$;
6. $\operatorname{ran} T = \{Tx \ : \ x \in X\}$ is the range of $T \in [X]$;
7. $\mathsf{Cyc}\, T$ denotes the set of cyclic subspaces of an operator $T \in [X]$ (see Definition 2.12);
8. $\{T\}'$ and $\{T\}''$ denote the commutant and the double commutant (or bicommutant) of an operator $T \in [X]$, respectively;
9. $\mathsf{Alg}\{T_1, \ldots, T_n\}$ stands for a weakly closed subalgebra of $[X]$ generated by $T_1, \ldots, T_n \in [X]$ and the identity $\mathbb{I}$;
10. $\mathsf{Lat}\, \mathcal{A}$ denotes the lattice of invariant subspaces of the algebra $\mathcal{A}$;
11. $\mathsf{Lat}\, T$ $(:= \mathsf{Lat}(\mathsf{Alg}\, T))$ and $\mathsf{HypLat}\, T$ $(:= \mathsf{Lat}(\{T\}'))$ denote the lattices of invariant and hyperinvariant subspaces of $T \in [X]$, respectively;
12. $\operatorname{span} E$ is the closed linear span of the set $E \subset X$;
13. $r * f$ stands for the convolution of functions $r,\ f \in L_1[0,1]$: $(r * f)(x) := \int_0^x r(x-t) f(t)\, dt$;
14. $\mathbb{Z}_+ := \{n \in \mathbb{Z} : \ n \geqslant 0\}$; $\mathbb{R}_+ := \{x \in \mathbb{R} : \ x \geqslant 0\}$.

As usual, $W_p^k[0,1]$ ($p \in (1,\infty),\ k \in \mathbb{Z}_+ \setminus \{0\}$) stands for the Sobolev space consisting of functions f having $k-1$ absolutely continuous derivatives and $f^{(k)} \in L_p[0,1]$. $W_p^k[0,1]$ is a Banach space equipped with the norm

$$\|f\|_{W_p^k[0,1]} = \left[\sum_{j=0}^{k-1} |f^{(j)}(0)|^p + \int_0^1 |f^{(k)}(t)|^p \, dt \right]^{1/p}.$$

$W^k_{p,0}[0,1] := \{f \in W^k_p[0,1] : \ f(0) = \cdots = f^{(k-1)}(0) = 0\}$.

We set $W^0_p[0,1] := L_p[0,1]$ and $W^0_{p,0}[0,1] = L_p[0,1]$.

Let $J^\alpha_{k,0}$ and $J^\alpha_k := J^\alpha_{k,k}$ stand for the operator J^α defined on $W^k_{p,0}[0,1]$ and $W^k_p[0,1]$, respectively. The operator $J^\alpha_{k,0}$ is well defined on $W^k_{p,0}[0,1]$ for any $\alpha > 0$. The operator J^α_k is well defined on $W^k_p[0,1]$ if either $\alpha \in \mathbb{Z}_+ \setminus \{0\}$ or $\alpha > k - \frac{1}{p}$. Therefore throughout the paper we assume that

1. the operator $A := \bigoplus_{j=1}^n \lambda_j J^\alpha$ is defined on $\bigoplus_{j=1}^n L_p[0,1]$ for $\alpha > 0$;
2. the operator $A_{k,0} := \bigoplus_{j=1}^n \lambda_j J^\alpha_{k_j,0}$ is defined on $\bigoplus_{j=1}^n W^{k_j}_{p,0}[0,1]$ with $k_j \geqslant 0$ and $\alpha > 0$;
3. the operator $A_k := \bigoplus_{j=1}^n \lambda_j J^\alpha_{k_j}$ is defined on $\bigoplus_{j=1}^n W^{k_j}_p[0,1]$ with $k_j \geqslant 1$ and for $\alpha \in \mathbb{Z}_+ \setminus \{0\}$ or $\alpha > \max\limits_{1 \leqslant j \leqslant n} k_j - \frac{1}{p}$.

We will also assume that $\lambda_j \neq 0$ for $j \in \{1, \ldots, n\}$.

2 Preliminaries

2.1 Invariant subspaces of some operators

Here we present some known results on invariant subspaces of finite-dimensional nilpotent operators and C_0 contractions.We also recall a condition about splitting of $\mathsf{Alg}(A \oplus B)$, where $A, B \in [X]$.

Theorem 2.1. *[7, 21] If Q is nilpotent on a finite-dimensional vector space V, then*

$$\mathsf{Lat}(Q) = \bigcup_M \left\{[M, Q^{-1}M] : \ M \in \mathsf{Lat}(Q \upharpoonright QV)\right\}, \tag{2.1}$$

where $[M, Q^{-1}M]$ is an interval in the lattice of all subspaces of V. Each interval satisfies the equation

$$\dim Q^{-1}M - \dim M = \dim \ker Q. \tag{2.2}$$

The following result was first discovered by P. Halmos [22] for operators defined on finite-dimensional spaces. The generalization to C_0 contractions on Hilbert spaces belongs to H. Bercovici [2, Proposition 5.33], [3, Corollary 2.11] and P. Wu [48, Theorem 1.2], and [49, Theorem 5])(see also references therein).

Theorem 2.2. *Let T be a C_0-contraction defined on a separable Hilbert space. Then every invariant subspace of T is the closure of the range and the kernel of some bounded linear transformation that commutes with T, that is,*

$$\mathsf{Lat}\, T = \left\{\ker C : \ C \in \{T\}'\right\} = \left\{\overline{\operatorname{ran} C} : \ C \in \{T\}'\right\}.$$

Definition 2.3. (see [33],[36]) Let A and B be bounded operators defined on a Banach space X_1 and X_2 respectively. A is said to be quasisimilar to B if there exist deformations $K : \ X_1 \to X_2$ and $L : \ X_2 \to X_1$ (i.e. $\overline{\operatorname{ran} K} = X_2$, $\ker K = \{0\}$, $\overline{\operatorname{ran} L} = X_1$, $\ker L = \{0\}$) such that $AL = LB$ and $KA = BK$.

Remark 2.4. (i) Standard manipulations with Cayley transform implies that Theorem 2.2 holds also for quasinilpotent accretive operators with finite-dimensional real part.

(ii) Let operator A be defined on a Banach space. Let also A be quasisimilar to a C_0 contraction T. Then, obviously the statement of Theorem 2.2 is true for A, that is, $\mathsf{Lat}\, A = \{\ker C : \ C \in \{A\}'\} = \{\overline{\operatorname{ran} C} : \ C \in \{A\}'\}$.

Let X be a Banach space and let n be a positive integer. Then $X^{(n)}$ denotes the direct sum of n copies of X. If A is an operator on X, then $A^{(n)}$ denotes the direct sum of n copies of A (regarded as an operator on $X^{(n)}$).

The following theorem is implicitly contained in [43] (see also [42, Theorem 7.1, Theorem 7.2])

Theorem 2.5. *Let $T_1, \dots, T_r \in [X]$ and*

$$\mathsf{Lat}(T_1^{(n)} \oplus \cdots \oplus T_r^{(n)}) = \mathsf{Lat}\, T_1^{(n)} \oplus \cdots \oplus \mathsf{Lat}\, T_r^{(n)}, \qquad n = 1, 2, \dots$$

Then $\mathsf{Alg}(T_1 \oplus \cdots \oplus T_r) = \mathsf{Alg}\, T_1 \oplus \cdots \oplus \mathsf{Alg}\, T_r$.

2.2 Spectral analysis of the operator $A = \bigoplus_{i=1}^n \lambda_i J^\alpha$ defined on $\bigoplus_{i=1}^n L_p[0,1]$

Throughout this subsection X stands for $L_p[0,1]$, with $p \in (1, \infty)$. Here we present some results from [31] on spectral analysis of the operator $A = \bigoplus_{i=1}^n \lambda_i J^\alpha$ defined on $\bigoplus_1^n X$. Moreover, we obtain a description of $\mathsf{Alg}\, A$ and investigate its properties.

We begin with the following simple statement.

Lemma 2.6. *Let $A_i, M_i, N_i \in [X]$ for $i \in \{1, \dots, n\}$ and $A = \bigoplus_{i=1}^n A_i$. Assume also that the following identities are satisfied*

$$A_i^m = M_i A_1^m N_i, \qquad m \in \mathbb{Z}_+, \qquad i \in \{1, \dots, n\}. \tag{2.3}$$

Then

$$\mathsf{Alg}\, A = \left\{ \bigoplus_{i=1}^n R_i : \quad R_1 \in \mathsf{Alg}\, A_1, \quad R_i = M_i R_1 N_i, \quad i \in \{2, \dots, n\} \right\}. \tag{2.4}$$

Proof. Let $M := \bigoplus_{i=1}^n M_i$ and $N := \bigoplus_{i=1}^n N_i$. Then for any polynomial $p(\cdot)$ identities (2.3) yield $p(A_i) = M_i p(A_1) N_i$. Hence,

$$p(A) = \bigoplus_{i=1}^n p(A_i) = \bigoplus_{i=1}^n M_i p(A_1) N_i = M \left(\bigoplus_{i=1}^n p(A_1) \right) N.$$

On the other hand, by definition of $\mathsf{Alg}\, A$ polynomials $p(A)$ are dense in $\mathsf{Alg}\, A$ in weak operator topology. Hence the last identities imply $\mathsf{Alg}\, A = M\, \mathsf{Alg}(\bigoplus_{i=1}^n A_1) N$. To complete the proof it remains to note that $\mathsf{Alg}(\bigoplus_{i=1}^n A_1) = \bigoplus_{i=1}^n \mathsf{Alg}(A_1)$. $\square$

Next we apply Lemma 2.6 to describe $\mathsf{Alg}\, A$ for the operator $A = \bigoplus_{i=1}^n \lambda_i J^\alpha$ with factors λ_i having equal arguments,

$$\lambda_i = \lambda_1 / s_i^\alpha \ , \quad 1 = s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n, \qquad i \in \{1, \ldots, n\}. \tag{2.5}$$

Theorem 2.7. *Let the operator* $A = \bigoplus_{i=1}^n \lambda_i J^\alpha$ *be defined on* $\bigoplus_{i=1}^n X$ *with* λ_i *satisfying condition* (2.5)*. Then* $\mathsf{Alg}\, A$ *is*

$$\begin{aligned} \mathsf{Alg}\, A = \Big\{ R = \operatorname{diag}(R_1, \ldots, R_n) : (R_i f)(x) = \frac{d}{dx} \int_0^x r_i(x-t) f(t)\, dt, \\ r_1 \in L_{p'}[0,1], \quad r_i(x) = r_1(s_i^{-1} x), \quad R_1 \in [L_p[0,1]] \Big\}. \end{aligned} \tag{2.6}$$

Proof. To apply Lemma 2.6 we introduce the operators M_i and N_i by setting

$$(M_i f)(x) := f(s_i^{-1} x), \quad (N_i f)(x) := \begin{cases} f(s_i x), & x \in [0, s_i^{-1}], \\ 0, & x \in [s_i^{-1}, 1]. \end{cases} \tag{2.7}$$

Clearly, $\ker N_i = \{0\}$, $\operatorname{ran} N_i = \chi_{[0, s_i^{-1}]} L_p[0,1]$, $\ker M_i = \chi_{[s_i^{-1}, 1]} L_p[0,1]$ and $\operatorname{ran} M_i = L_p[0,1]$. It can easily be checked that $M_i N_i = I_{L_p[0,1]}$ and, moreover,

$$(\lambda_i J^\alpha)^m = M_i (\lambda_1 J^\alpha)^m N_i, \qquad m \in \mathbb{Z}_+, \qquad i \in \{1, \ldots, n\}.$$

Setting $A_i := \lambda_i J^\alpha$ and applying Lemma 2.6 we obtain

$$\mathsf{Alg}\, A = \Big\{ R = \bigoplus_{i=1}^n R_i : R_1 \in \mathsf{Alg}(\lambda_1 J^\alpha), \quad R_i = M_i R_1 N_i, \quad i \in \{2, \ldots, n\} \Big\}. \tag{2.8}$$

On the other hand, according to (1.3), any (bounded) $R_1 \in \mathsf{Alg}(\lambda_1 J^\alpha)$ admits a representation

$$R_1 \ : \ f(x) \to \frac{d}{dx} \int_0^x r_1(x-t) f(t)\, dt, \qquad r_1 \in L_{p'}[0,1]. \tag{2.9}$$

Straightforward calculations show that that

$$(M_i R_1 N_i f)(x) = \frac{d}{dx} \int_0^x r_1(s_i^{-1}(x-t)) f(t)\, dt, \qquad i \in \{2, \ldots, n\}.$$

Combining the last equality with (2.8) we complete the proof. $\square$

To state the results on $\{A\}'$ we need some additional notations. For any $a \in \mathbb{R}_+ \backslash \{0\}$ we define an operator $L_a : \ X \to X$ by

$$L_a : \ f(x) \to g(x) = \begin{cases} f(ax), & 0 < a \leqslant 1, \\ \begin{cases} 0, & x \in [0, 1 - a^{-1}], \\ f(ax - a + 1), & x \in [1 - a^{-1}, 1], \end{cases} & a > 1. \end{cases} \tag{2.10}$$

We set also

$$L_a \{J^\alpha\}' := \{L_a K : \ K \in \{J^\alpha\}'\}, \qquad \{J^\alpha\}' L_a := \{K L_a : \ K \in \{J^\alpha\}'\}.$$

It is easily checked that $L_a \{J^\alpha\}' = \{J^\alpha\}' L_a$.

Theorem 2.8. *[31, Proposition 4.6] Suppose $A = \bigoplus_{i=1}^n \lambda_i J^\alpha$ is defined on $\bigoplus_{i=1}^n X$ and λ_i satisfy condition* (2.5). *Set also $a_{ij} := s_i^{-1} s_j$ for $i, j \in \{1, \dots, n\}$. Then the commutant $\{A\}'$ is of the form*

$$\{A\}' = \{K : \ K = (K_{ij})_{i,j=1}^n, \quad K_{ij} \in L_{a_{ij}} \{J^\alpha\}'\}.$$

Next we complete Theorem 2.7 by establishing the Neumann type identity, $\{A\}'' = \mathsf{Alg}\, A$. Note, that for the case $p = 2$ and $\alpha = 1$ it follows from a general result of B.S.-Nagy and C. Foias [34] on a dissipative operator with finite dimensional imaginary part.

Theorem 2.9. *Suppose $A = \bigoplus_{i=1}^n \lambda_i J^\alpha$ is defined on $\bigoplus_{i=1}^n X$ and λ_i satisfy condition* (2.5). *Then $\{A\}'' = \mathsf{Alg}\, A$.*

Proof. It is known (and easily seen) that if T_1 and T_2 are bounded operators on a Banach space Y, then $\{T_1 \oplus T_2\}'' \subset \{T_1\}'' \oplus \{T_2\}''$. Hence $\{A\}'' = \{\bigoplus_{i=1}^n \lambda_i J^\alpha\}'' \subset \bigoplus_{i=1}^n \{\lambda_i J^\alpha\}''$. It follows that any $R \in \{A\}''$ admits a direct sum decomposition $R = \bigoplus_{i=1}^n R_i$ with $R_i \in \{\lambda_i J^\alpha\}'' = \{\lambda_i J^\alpha\}'$, $i \in \{1, ..., n\}$. According to (1.3) R_i admits a representation $(R_i f)(x) = \frac{d}{dx} \int_0^x r_i(x-t) f(t)\, dt$, where $r_i \in L_{p'}[0,1]$ and it is such that $R_i \in [X]$.

Further, let $K = (K_{ij})_{i,j=1}^n$ be an operator matrix with entries $K_{ij} = L_{a_{ij}}$ for $i > j$ and $K_{ij} = \mathbb{O}$ for $i \le j$. Let also $a_{ij} := s_i^{-1} s_j$ for $i, j \in \{1, \dots, n\}$. Then, by Theorem 2.8, $K \in \{A\}'$. Clearly, relation $RK = KR$ yields

$$R_i L_{a_{i1}} = L_{a_{i1}} R_1, \qquad i \in \{2, \dots, n\}. \tag{2.11}$$

It is easily seen that

$$(R_i L_{a_{i1}} f)(x) = \frac{d}{dx} \int_0^x r_i(x-t) f(s_i^{-1} t)\, dt, \quad i \in \{2, \dots, n\}. \tag{2.12}$$

On the other hand,

$$\begin{aligned}(L_{a_{i1}} R_1 f)(x) =& \frac{d}{dx_1} \int_0^{x_1} r_1(x_1 - t) f(t)\, dt \Big|_{x_1 = s_i^{-1} x} \\ =& \, s_i \frac{d}{dx} \int_0^{s_i^{-1} x} r_1(s_i^{-1} x - t) f(t)\, dt = \frac{d}{dx} \int_0^x r_1(s_i^{-1}(x-t)) f(s_i^{-1} t)\, dt.\end{aligned}$$

Comparing this relation with (2.12) and taking into account (2.11) and the obvious relation $\operatorname{ran}(L_{a_{i1}}) = X$, we obtain $r_i(x) = r_1(s_i^{-1} x)$, $i \in \{2, \dots, n\}$. By Theorem 2.7, this means that $R \in \mathsf{Alg}\, A$, that is $\{A\}'' \subset \mathsf{Alg}\, A$. Since the inclusion $\{A\}'' \supset \mathsf{Alg}\, A$ is obvious, we get $\{A\}'' = \mathsf{Alg}\, A$. □

In the following theorem we obtain a description of $\mathsf{Lat}\, A$ similar to that of $\mathsf{Lat}\, T$ for C_0-contractions T described in Theorem 2.2. It is interesting to note that though a description is completely the same, the operator A in not accretive in $L_2[0,1]$ for $\alpha > 1$ (cf. Remark 2.4 (i)).

Theorem 2.10. *Let $A = \bigoplus_{i=1}^n \lambda_i J^\alpha$ be defined on $\bigoplus_{i=1}^n X$ and λ_i satisfy conditions* (2.5). *Then every invariant subspace of A is the closure of the range (the kernel) of a bounded linear transformation that commutes with A.*

Proof. Alongside the operator A we consider the operator $A_1 := \bigoplus_{i=1}^n \lambda_1 s_i^{-1} J$. By Theorem 2.7, $\mathsf{Alg}\, A = \mathsf{Alg}(\bigoplus_{i=1}^n \lambda_1 s_i^{-\alpha} J^\alpha) = \mathsf{Alg}(\bigoplus_{i=1}^n \lambda_1 s_i^{-1} J) = \mathsf{Alg}\, A_1$. Hence $\mathsf{Lat}\, A = \mathsf{Lat}\, A_1$ and $\{A\}' = \{A_1\}'$. So we can assume that $\lambda_1 = 1$ and $\alpha = 1$. We put

$$K := \bigoplus_{i=1}^n J \in \left[\bigoplus_{i=1}^n L_p[0,1], \bigoplus_{i=1}^n L_2[0,1]\right],$$

$$L := \bigoplus_{i=1}^n J \in \left[\bigoplus_{i=1}^n L_2[0,1], \bigoplus_{i=1}^n L_p[0,1]\right],$$

$$B := \bigoplus_{i=1}^n s_i J \in \left[\bigoplus_{i=1}^n L_2[0,1]\right].$$

It is clear that $\ker K = \{0\}$, $\ker L = \{0\}$, $\overline{\operatorname{ran} K} = \bigoplus_{i=1}^n L_2[0,1]$, $\overline{\operatorname{ran} L} = \bigoplus_{i=1}^n L_p[0,1]$, $KA_1 = BK$ and $A_1 L = LB$. Hence A_1 is quasisimilar to B. So, we can assume that A_1 is defined on $\bigoplus_{i=1}^n L_2[0,1]$. Note that A_1 is accretive, since $s_i > 0$ for $i \in \{1, ..., n\}$. Now the assertions of the theorem follow from Theorem 2.2 (see also Remark 2.4 (i)). □

Next, we recall a description of $\mathsf{HypLat}\, A$.

Theorem 2.11. *[31, Proposition 4.8] Suppose $A = \bigoplus_{i=1}^n \lambda_i J^\alpha$ is defined on $\bigoplus_{i=1}^n X$ and λ_i satisfy condition* (2.5). *Then the lattice* $\mathsf{HypLat}\, A$ *is of the form*

$$\mathsf{HypLat}\, A = \left\{\bigoplus_{i=1}^n E_{a_i} : \ (a_1, \dots, a_n) \in P(s_1, \dots, s_n)\right\},$$

where

$$P(s_1, \dots, s_n) := \big\{(a_1, \dots, a_n) \in [0,1]^n : \\ s_i a_{i+1} \leqslant s_{i+1} a_i \leqslant s_{i+1} - s_i + s_i a_{i+1}, \ 1 \leqslant i \leqslant n-1\big\}.$$

Definition 2.12. (cf. [36])

(1) A subspace E of a Banach space X_1 is called a cyclic subspace for an operator $T \in [X_1]$ if $\operatorname{span}\{T^n E : \ n \geqslant 0\} = X_1$;

(2) a vector $f (\in X_1)$ is called cyclic for T if $\operatorname{span}\{T^n f : \ n \geqslant 0\} = X_1$;

(3) the set of all cyclic subspaces of an operator T is denoted by $\mathsf{Cyc}\, T$.

Definition 2.13. (1) The number

$$\mu_T := \inf_E \{\dim E : \ E \text{ is a cyclic subspace of the operator } T \text{ on } X_1\}$$

is called the spectral multiplicity of an operator T on X_1;

(2) operator T is called cyclic if $\mu_T = 1$.

It is well known that the concept of spectral multiplicity plays an important role in control theory (see for instance [47]). Investigating some other problems of control theory, N.K. Nikol'skii and V.I. Vasjunin [38] introduced one more "cyclic" characteristic of an operator.

Definition 2.14. [38] Let $T \in [X]$. Then

$$\operatorname{disc} T := \sup_{E \in \mathsf{Cyc}\, T} \min\{\dim E' : \; E' \subset E, E' \in \mathsf{Cyc}\, T\}.$$

$\operatorname{disc} T$ is called a disc-characteristic of an operator T. ("disc" is the abbreviation of "Dimension of the Input Subspace of Control".)

Clearly, $\operatorname{disc} T \geqslant \mu_T$.

To present a description of $\mathsf{Cyc}\, A$ we recall the following definition.

Definition 2.15. [29, 31, 35]) The determinant of a functional matrix $F(x) = (f_{ij}(x))_{i,j=1}^n$ ($f_{ij} \in X$) calculated with respect to the convolution product

$$(f * g)(x) = \int_0^x f(x-t)g(t)\,dt = \int_0^x g(x-t)f(t)\,dt = (g * f)(x)$$

is called $*$ - determinant and is denoted by $* - \det F(x)$. Similarly, $*$ - minors of $F(x)$ are the minors calculated with respect to the convolution product. $*$-rank $F(x)$ will is the highest order of $*$-minors of $F(x)$ satisfying ε-condition (1.2).

Next we complete [31, Theorem 2.3] by computing $\operatorname{disc} A$.

Theorem 2.16. *Suppose $A = \bigoplus_{i=1}^n \lambda_i J^\alpha$ is defined on $\bigoplus_{i=1}^n X$ and λ_i satisfy condition* (2.5). *Then the system $\{f_l\}_{l=1}^N$ of vectors*

$$f_l = f_{l1} \oplus \cdots \oplus f_{ln} \in \bigoplus_{i=1}^n X, \qquad\qquad l \in \{1, \dots, N\}, \quad i \in \{1, \dots, n\}$$

generates a cyclic subspace for the operator A if and only if

(i) $N \geqslant n$;

(ii) the matrix

$$F_n(x) = \begin{pmatrix} f_{11}(s_1 x) & f_{12}(s_2 x) & \dots & f_{1n}(s_n x) \\ \vdots & \vdots & & \vdots \\ f_{N1}(s_1 x) & f_{N2}(s_2 x) & \dots & f_{Nn}(s_n x) \end{pmatrix}$$

is of maximal $*$-rank , *namely,* $*$-rank $F_n(x) = n$;

(iii) $\operatorname{disc} A = \mu_A = n$.

Proof. (i), (ii) and the equality $\mu_A = n$ were proved in [31, Theorem 2.3](see also [14, Proposition 3.2] for another proof).

(iii) Let us prove that $\operatorname{disc} A = n$. Let $E = \operatorname{span}\{f_1, \dots, f_N\}$ be an N-dimensional subspace cyclic for the operator A. It is necessary to show that this space contains an n-dimensional subspace which is also cyclic for the operator A. Since $*\text{-rank}\, F_n(x) = n$, it follows that there exists an $n \times n$ submatrix $G_n(x)$ of $F_n(x)$ such that $*\text{-rank}\, G_n(x) = n$. Hence we can choose n- vectors $f_{i_1}, \dots, f_{i_n}$ $(i_1, \dots, i_n \in \{1, \dots, N\})$ such that $\operatorname{span}\{f_{i_1}, \dots, f_{i_n}\}$ is a cyclic subspace for A. $\square$

Corollary 2.17. *Let $K \in \{J^\alpha\}'$ and $K_n = \bigoplus_{i=1}^n K$ be defined on $\bigoplus_{i=1}^n L_2[0,1]$. Then $\mu_{K_n} \geq n$.*

Proof. It follows from Theorem 2.7 that $K_n \in \mathsf{Alg}\, A$, where $A = \bigoplus_{i=1}^n J$ is defined $\bigoplus_{i=1}^n L_2[0,1]$. Hence, by Theorem 2.16 $\mu_{K_n} \geq \mu_A = n$. $\square$

Remark 2.18. In the recent paper [4, Proposition 7.6] Corollary 2.17 was proved for the case $n = 2$.

Next we recall the following notation. Let $T_j \in [X_j]$ $(j = 1, 2)$ and $R \in \mathsf{Cyc}(T_1 \oplus T_2)$. It is clear that $P_j R \in \mathsf{Cyc}\, T_j$, where P_j is the projection from $X_1 \oplus X_2$ onto X_j, $j \in \{1, 2\}$. Following [38], we write

$$\mathsf{Cyc}(T_1 \oplus T_2) = \mathsf{Cyc}\, T_1 \vee \mathsf{Cyc}\, T_2$$

if $P_j R \in \mathsf{Cyc}\, T_j$ $(j = 1, 2)$ yields $R \in \mathsf{Cyc}(T_1 \oplus T_2)$ for every $R \subset X_1 \oplus X_2$. In particular, if $\mathsf{Lat}(T_1 \oplus T_2) = \mathsf{Lat}\, T_1 \oplus \mathsf{Lat}\, T_2$ then $\mathsf{Cyc}(T_1 \oplus T_2) = \mathsf{Cyc}\, T_1 \vee \mathsf{Cyc}\, T_2$.

Next we complete [31, Proposition 4.1, Proposition 4.2].

Theorem 2.19. *Suppose $A = \bigoplus_{j=1}^r \lambda_j J^\alpha$ is defined on $\bigoplus_{j=1}^r X$ and*

$$\arg \lambda_i \neq \arg \lambda_j \pmod{2\pi}, \qquad 1 \leqslant i < j \leqslant r. \tag{2.13}$$

Then

$$\mathsf{Alg}\, A = \{A\}' = \{A\}'' = \bigoplus_{j=1}^r \mathsf{Alg}\, J^\alpha = \bigoplus_{j=1}^r \{J^\alpha\}' = \bigoplus_{j=1}^r \{J^\alpha\}'', \tag{2.14}$$

$$\mathsf{Lat}\, A = \mathsf{HypLat}\, A = \bigoplus_{j=1}^r \mathsf{Lat}\, J^\alpha = \bigoplus_{j=1}^r \mathsf{HypLat}\, J^\alpha, \tag{2.15}$$

$$\mathsf{Cyc}\, A = \bigvee_{j=1}^r \mathsf{Cyc}\, J^\alpha, \tag{2.16}$$

$$\operatorname{disc} A = \mu_A = 1. \tag{2.17}$$

Proof. (2.15)-(2.17) and the splitting of $\{A\}'$ and $\{A\}''$ were proved in [29], [31]. We present two different proofs of the splitting of $\mathsf{Alg}\, A$ due to the first and to the second author, respectively.

First proof. We will derive the splitting of $\mathsf{Alg}\, A$ from the splitting of $\mathsf{Cyc}\, A$.

By (2.16) $g := \frac{x^{\alpha-1}}{\Gamma(\alpha)} \oplus \cdots \oplus \frac{x^{\alpha-1}}{\Gamma(\alpha)} \in \mathsf{Cyc}\, A$. Hence, there exists a sequence $\{P_n(x)\}_{n=1}^{\infty}$ such that $\text{s-lim}_{n\to\infty} P_n(A)g = 0 \oplus \cdots \oplus 0 \oplus \frac{x^{\alpha-1}}{\Gamma(\alpha)}$. We claim that

$$\text{s-lim}_{n\to\infty} AP_n(A) = \mathbb{O} \oplus \cdots \oplus \mathbb{O} \oplus \lambda_r J^{\alpha}. \tag{2.18}$$

Indeed, for any $f = f_1 \oplus \cdots \oplus f_r \in \bigoplus_{j=1}^{r} X$ one has

$$\begin{aligned}
&\text{s-lim}_{n\to\infty} AP_n(A)f = \text{s-lim}_{n\to\infty} (\lambda_1 J^{\alpha} P_n(\lambda_1 J^{\alpha}) f_1 \oplus \cdots \oplus \lambda_r J^{\alpha} P_n(\lambda_r J^{\alpha}) f_r) \\
&= \text{s-lim}_{n\to\infty} \left(\lambda_1 \frac{x^{\alpha-1}}{\Gamma(\alpha)} * (P_n(\lambda_1 J^{\alpha}) f_1)(x) \oplus \cdots \oplus \lambda_r \frac{x^{\alpha-1}}{\Gamma(\alpha)} * (P_n(\lambda_r J^{\alpha}) f_r)(x) \right) \\
&= \text{s-lim}_{n\to\infty} \left(\lambda_1 f_1 * (P_n(\lambda_1 J^{\alpha}) \frac{x^{\alpha-1}}{\Gamma(\alpha)}) \oplus \cdots \oplus, \lambda_r f_r * (P_n(\lambda_r J^{\alpha}) \frac{x^{\alpha-1}}{\Gamma(\alpha)} \right) \\
&= \lambda_1 f_1 * 0 \oplus \lambda_2 f_2 * 0 \oplus \cdots \oplus \lambda_r f_r * \frac{x^{\alpha-1}}{\Gamma(\alpha)} = \operatorname{diag}(\mathbb{O}, \ldots, \mathbb{O}, \lambda_r J^{\alpha}) f.
\end{aligned}$$

So (2.18) is proved. A similar argument shows that for any $j \in \{1, \ldots, r\}$ there exists a sequence of polynomials $\{P_{j,n}\}_{n=1}^{\infty}$ such that

$$\text{s-lim}_{n\to\infty} AP_{j,n}(A) = \mathbb{O} \oplus \cdots \oplus \mathbb{O} \oplus \lambda_j J^{\alpha} \oplus \mathbb{O} \oplus \cdots \oplus \mathbb{O}.$$

Hence the splitting of $\mathsf{Alg}\, A$ is proved.

Second proof. Keeping in mind notations of Theorem 2.24 (see below), for any $j \in \{1, \ldots, r\}$ we let $n_j = n$ and $\lambda_{j1} := \cdots := \lambda_{jn} := \lambda_j$. Then setting $A(j) := \bigoplus_{i=1}^{n_j} \lambda_{ji} J^{\alpha}$ we rewrite $A(j)$ and A as

$$A(j) = \bigoplus_{i=1}^{n} \lambda_j J^{\alpha} = (\lambda_j J^{\alpha})^{(n)} \quad \text{and} \quad A = \bigoplus_{j=1}^{r} A(j) = \bigoplus_{j=1}^{r} (\lambda_j J^{\alpha})^{(n)},$$

where the factors λ_j have different arguments, $\lambda_j \neq \lambda_k$ for $j \neq k$. Therefore by Theorem 2.24 the lattice $\mathsf{Lat}(\bigoplus_{j=1}^{r} (\lambda_j J^{\alpha})^{(n)})$ splits, $\mathsf{Lat}(\bigoplus_{j=1}^{r} (\lambda_j J^{\alpha})^{(n)}) = \bigoplus_{j=1}^{r} \mathsf{Lat}(\lambda_j J^{\alpha})^{(n)}$. One completes the proof by applying Theorem 2.5 with $T_j = \lambda_j J^{\alpha},\ j \in \{1, \ldots, n\}$.
$\square$

Remark 2.20. Some particular statements of Theorem 2.19 were obtained in [1, 23, 39, 40] for the case $p = 2$.

Namely, A. Atzmon [1] proved that for every integer $k \geq 2$, the operator $iJ^{1-1/k} \oplus e^{\frac{\pi i}{2k}} J^{1-1/k}$ is cyclic.

In [39, 40] B.P. Osilenker and V.S. Shulman proved that (2.13) implies the splitting of $\mathsf{Lat}(\bigoplus_{j=1}^{r} \lambda_j J)$. Their proof cannot be extended to the case $\alpha \neq 1$.

L.T. Hill [23] showed that if $\alpha \in (0, 1)$ and λ is a nonzero complex number, then $\mathsf{Lat}(J^{\alpha} \oplus \lambda J^{\alpha})$ splits if and only if λ is not positive. His proof cannot be extended neither to the case of $\alpha > 1$ nor to the number of summands $n > 2$.

The following result is easily implied by combining Theorems 2.8 and 2.19.

Corollary 2.21. *[28],[31] Let $c \in \mathbb{C}$ and let $R \in [X]$ be a solution of the equation $RJ^{\alpha} = cJ^{\alpha}R$. Then the following statements hold*

(i) *if* $c \notin \mathbb{R}_+$, *then* $R = \mathbb{O}$;

(ii) *if* $c = a^\alpha > 0$, $a > 0$, *then* $R \in L_a\{J^\alpha\}'$, *where* L_a *is defined by* (2.10).

Remark 2.22. (i) It was shown in [19] that the operators J and cJ are similar if and only if $c = 1$. Corollary 2.21 implies that operators J^α and cJ^α are not even quasisimilar for any $c \neq 1$.

(ii) In particular cases Corollary 2.21 (i) was recently reproved by another method in [5], [26] (the case $\alpha = 1$, $p = 2$) and in [6] (the case $\alpha \in \mathbb{Z}_+ \setminus \{0\}$, $p = 2$). Some solutions R of the equation $RJ^\alpha = cJ^\alpha R$ in the case $c > 0$, $\alpha \in \mathbb{Z}_+$ were also indicated in [5], [6], [26].

We need the following lemma in the sequel.

Lemma 2.23. *Suppose that* $A \in [X_1]$ *is quasisimilar to* $B \in [X_2]$ *with intertwining deformations* L *and* K. *That is,* $AL = LB$ *and* $KA = BK$. *Let also* $LK = A^2$ *and* $KL = B^2$. *Then*

(i) $E \in \mathsf{Cyc}\, A \Leftrightarrow \overline{KE} \in \mathsf{Cyc}\, B$;

(ii) $F \in \mathsf{Cyc}\, B \Leftrightarrow \overline{LF} \in \mathsf{Cyc}\, A$;

(iii) $\operatorname{disc} A = \operatorname{disc} B$.

Proof. The proof is left for the reader. □

Now we can consider the case of any diagonal nonsingular matrix B.

Next we complete [31, Proposition 3.2, Theorem 3.4, Corollary 3.5, Theorem 4.10, Theorem 4.11].

Theorem 2.24. *Suppose* $A(j) := \bigoplus_{i=1}^{n_j} \lambda_{ji} J^\alpha$ *is defined on* $\bigoplus_{i=1}^{n_j} X$, $j \in \{1, \ldots, r\}$ *and* $A := \bigoplus_{j=1}^{r} A(j)$ *is defined on* $\bigoplus_{j=1}^{r}(\bigoplus_{i=1}^{n_j} X)$. *Let also*

$$\begin{aligned} \arg \lambda_{j1} &= \arg \lambda_{ji} \pmod{2\pi}, \qquad && 1 \leqslant j \leqslant r, \quad 1 \leqslant i \leqslant n_j, \\ \arg \lambda_{i1} &\neq \arg \lambda_{j1} \pmod{2\pi}, \qquad && 1 \leqslant i < j \leqslant r. \end{aligned}$$

Then

$$\mathsf{Alg}\,A=\bigoplus_{j=1}^{r}\mathsf{Alg}\,A(j), \tag{2.19}$$

$$\{A\}'=\bigoplus_{j=1}^{r}\{A(j)\}', \tag{2.20}$$

$$\{A\}''=\bigoplus_{j=1}^{r}\{A(j)\}'', \tag{2.21}$$

$$\mathsf{Lat}\,A=\bigoplus_{j=1}^{r}\mathsf{Lat}\,A(j), \tag{2.22}$$

$$\mathsf{HypLat}\,A=\bigoplus_{j=1}^{r}\mathsf{HypLat}\,A(j), \tag{2.23}$$

$$\mathsf{Cyc}\,A=\bigvee_{j=1}^{r}\mathsf{Cyc}\,A(j), \tag{2.24}$$

$$\operatorname{disc}A=\mu_A=\max_{1\leqslant j\leqslant r}\mu_{A(j)}.$$

Proof. Relations (2.20)-(2.24) and the equality $\mu_A=\max\limits_{1\leqslant j\leqslant r}\mu_{A(j)}$ were proved in [31]. Let us prove (2.19). By Theorem 2.19, for any $j\in\{1,\dots,r\}$

$$\mathbb{O}\oplus\dots\oplus\mathbb{O}\oplus\lambda_{j1}J^\alpha\oplus\mathbb{O}\oplus\dots\oplus\mathbb{O}\in\mathsf{Alg}(\lambda_{11}J^\alpha\oplus\dots\oplus\lambda_{r1}J^\alpha).$$

Thus, by Theorem 2.7 we have that

$$\mathbb{O}\oplus\dots\oplus\mathbb{O}\oplus A(j)\oplus\mathbb{O}\oplus\dots\oplus\mathbb{O}\in\mathsf{Alg}\,A,$$

and hence (2.19) is proved.

Let us prove that $\operatorname{disc}A=\mu_A$. Assume that $A_1:=A$ is defined on $\bigoplus_{j=1}^{r}(\bigoplus_{i=1}^{n_j}L_2[0,1])$. Then [37, Statement 1.13] and [38, Corollary 13] imply the equality $\operatorname{disc}A_1=\mu_{A_1}$. We define

$$K:=\bigoplus_{j=1}^{r}\bigoplus_{i=1}^{n_j}\lambda_{ji}J^\alpha\in\left[\bigoplus_{j=1}^{r}\Big(\bigoplus_{i=1}^{n_j}L_2[0,1]\Big),\bigoplus_{j=1}^{r}\Big(\bigoplus_{i=1}^{n_j}L_p[0,1]\Big)\right],$$

$$L:=\bigoplus_{j=1}^{r}\bigoplus_{i=1}^{n_j}\lambda_{ji}J^\alpha\in\left[\bigoplus_{j=1}^{r}\Big(\bigoplus_{i=1}^{n_j}L_p[0,1]\Big),\bigoplus_{j=1}^{r}\Big(\bigoplus_{i=1}^{n_j}L_2[0,1]\Big)\right],$$

and $A_2:=A$. It is clear that K and L are deformations and $A_1L=LA_2$, $KA_1=A_2K$. Now application of Lemma 2.23 completes the proof. $\square$

3 The operator $A_{k,0}$

Let $J^\alpha_{k,l}$ stand for the operator J^α_k acting on the subspace E^k_l of $W^k_p[0,1]$ defined by (1.10) ($l\leqslant k-1$) and $E^k_k:=W^k_p[0,1]$.

Next we establish isometric equivalence of $J^\alpha_{k,0}$ and J^α.

Lemma 3.1. *The operator $J_{k,l}^{\alpha}$ defined on E_l^k is isometrically equivalent to the operator J_l^{α} defined on $W_p^l[0,1]$. In particular, the operator $J_{k,0}^{\alpha}$ defined on $W_{p,0}^k[0,1]$ is isometrically equivalent to the operator $J_0^{\alpha} =: J^{\alpha}$ defined on $W_p^0[0,1] = L_p[0,1]$.*

Proof. It is clear that the operator $U = \frac{d^{k-l}}{dx^{k-l}} : \ E_l^k \to W_p^l[0,1]$ isometrically maps E_l^k on $W_p^l[0,1]$. Moreover,

$$U^{-1} = U^* = J^{k-l} : \ W_p^l[0,1] \to E_l^k.$$

The assertion follows now from the identity $J_{k,l}^{\alpha} = U^{-1} J_l^{\alpha} U$. $\square$

Corollary 3.2. *The operator $A_{k,0} := \bigoplus_{i=1}^n \lambda_i J_{k_i,0}^{\alpha}$ defined on $\bigoplus_{i=1}^n W_{p,0}^{k_i}[0,1]$ is isometrically equivalent to the operator $A := \bigoplus_{i=1}^n \lambda_i J^{\alpha}$ defined on $\bigoplus_{i=1}^n L_p[0,1]$.*

Corollary 3.2 makes it possible to translate all results on the operator A defined on $\bigoplus_{i=1}^n L_p[0,1]$ to the results on operator $A_{k,0}$ defined on $\bigoplus_{i=1}^n W_{p,0}^{k_i}[0,1]$. For instance, Theorem 2.10 takes the following form

Theorem 3.3. *Let $A_{k,0} := \bigoplus_{i=1}^n \lambda_i J_{k_i,0}^{\alpha}$ be defined on $\bigoplus_{i=1}^n W_{p,0}^{k_i}[0,1]$ and λ_i satisfy condition (2.5). Then every invariant subspace of $A_{k,0}$ is the closure of the range (the kernel) of a bounded linear transformation that commutes with $A_{k,0}$.*

4 The operator A_k

This section contains the main results of the paper. Namely, we described the spectral properties of the operator $A_k := \bigoplus_{j=1}^n \lambda_j J_k^{\alpha}$ defined on $X^{(n)} = \bigoplus_1^n X$ where $X = W_p^k[0,1]$.

4.1 The algebra $\mathsf{Alg}\, A_k$

Theorem 4.1. *Suppose $A_k = \bigoplus_{i=1}^n \lambda_i J_k^{\alpha}$ is defined on $\bigoplus_{i=1}^n W_p^k[0,1]$ and*

$$\lambda_i = \lambda_1 / s_i^{\alpha}, \quad 1 = s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n, \qquad i \in \{1, \ldots, n\}. \tag{4.1}$$

Let also

$$R := \bigoplus_{i=1}^n R_i \in \left[\bigoplus_{i=1}^n W_p^k[0,1]\right], \qquad (R_i f)(\cdot) = c_i f(\cdot) + (r_i * f)(\cdot), \quad i \in \{1, \ldots, n\}. \tag{4.2}$$

Then the following is true:

(1) if $1 \leqslant \alpha \leqslant k-1$, then

$$\begin{aligned} \mathsf{Alg}\, A_k = \big\{ R : \ & c_1 = \cdots = c_n \in \mathbb{C}; \ r_1 \in W_p^{k-1}[0,1]; \ r_i(x) = s_i^{-1} r_1(s_i^{-1} x), \\ & 1 \leqslant i \leqslant n; \ r_1^{(l)}(0) = 0, \quad l \neq m\alpha - 1, \quad 1 \leqslant m \leqslant [(k-1)/\alpha] \big\}; \end{aligned} \tag{4.3}$$

(2) if $2 \leqslant k \leqslant \alpha + \frac{1}{p}$, then

$$\begin{aligned} \mathsf{Alg}\, A_k = \big\{R :\ c_1 = \cdots = c_n \in \mathbb{C};\ r_1 \in W^{k-1}_{p,0}[0,1]; \\ r_i(x) := s_i^{-1} r_1(s_i^{-1}x), \quad 1 \leqslant i \leqslant n\big\}. \end{aligned} \tag{4.4}$$

Proof. Let

$$(M_i f)(x) := f(s_i^{-1}x), \quad (N_i f)(x) := \begin{cases} f(s_i x), & x \in [0, s_i^{-1}], \\ \sum\limits_{m=0}^{k-1} \frac{(xs_i - 1)^m}{m!} f^{(m)}(1), & x \in [s_i^{-1}, 1]. \end{cases}$$

It can easily be checked that

$$(\lambda_i J_k^\alpha)^m = M_i (\lambda_1 J_k^\alpha)^m N_i, \qquad m \in \mathbb{Z}_+, \qquad i \in \{1, \ldots, n\}.$$

Setting $A_i := \lambda_i J_k^\alpha$ and applying Lemma 2.4 we obtain

$$\mathsf{Alg}\, A = \Big\{R = \bigoplus_{i=1}^n R_i : R_1 \in \mathsf{Alg}(\lambda_1 J_k^\alpha), \quad R_i = M_i R_1 N_i, \quad i \in \{2, \ldots, n\}\Big\}. \tag{4.5}$$

Next we confine ourselves to the case $1 \leqslant \alpha \leqslant k-1$. The case $2 \leqslant k \leqslant \alpha + \frac{1}{p}$ is considered similarly. By (1.12), $R_1 \in \mathsf{Alg}(\lambda_1 J_k^\alpha)$ if and only if

$$\begin{aligned} R_1\ :\ f(x) \to c_1 f(x) + \int_0^x r_1(x-t) f(t)\, dt, \quad c_1 \in \mathbb{C}, \quad r_1 \in W_p^{k-1}[0,1], \\ r_1^{(l)}(0) = 0, \quad l \neq m\alpha - 1, \quad 1 \leqslant m \leqslant [(k-1)/\alpha]. \end{aligned} \tag{4.6}$$

Straightforward calculations show that

$$(M_i R_1 N_i f)(x) = c_1 + \int_0^x s_i^{-1} r_1(s_i^{-1}(x-t)) f(t)\, dt, \qquad i \in \{2, \ldots, n\}.$$

Combining the last relations with (4.5) we arrive at the required description. $\square$

In the proof of the following theorem we need a concept of the weak operator topology in the algebra $[X]$. Recall the following definition.

Definition 4.2. Let $\{f_i\}_{i=1}^N)$ and $\{g_i\}_{i=1}^N$ be the sets of unit vectors in X and X^*, respectively, and let ε be a positive number. For any $R \in B[X]$ define $\mathcal{V} := \mathcal{V}(\varepsilon; \{f_i, g_i\}_{i=1}^N)$ to be the set of all operators T satisfying

$$|((T-R)f_i, g_i)| < \varepsilon, \qquad i \in \{1, \ldots, N\}.$$

Then $\mathcal{V}$ is a weak neighborhood of R and the family of all such sets $\mathcal{V}$ is a base of weak neighborhoods of R.

Theorem 4.3. *Suppose $A_k = \bigoplus_{j=1}^r \lambda_j J_k^\alpha$ is defined on $\bigoplus_{j=1}^r W_p^k[0,1]$ and*

$$\arg\lambda_i \neq \arg\lambda_j \pmod{2\pi}, \qquad 1 \leqslant i < j \leqslant r.$$

Let also

$$R := \bigoplus_{j=1}^r R_j \in \Big[\bigoplus_{j=1}^r W_p^k[0,1]\Big], \qquad (R_j f)(\cdot) = c_j f(\cdot) + (r_j * f)(\cdot), \quad j \in \{1,\dots,r\}.$$

Then the following are true:

(1) if $1 \leqslant \alpha \leqslant k-1$, then

$$\begin{aligned} \mathsf{Alg}\, A_k = \Big\{ R:\ & c_1 = \dots = c_r \in \mathbb{C};\ r_j \in W_p^{k-1}[0,1], \\ & r_j^{(\alpha m-1)}(0) = (\lambda_j\lambda_1^{-1})^m r_1^{(\alpha m-1)}(0), \quad m \leqslant \Big[\frac{k-1}{\alpha}\Big], \quad 1 \leqslant j \leqslant r; \\ & r_j^{(l)}(0) = 0, \quad l \neq \alpha m - 1, \quad m \leqslant [(k-1)/\alpha],\ 1 \leqslant j \leqslant r \Big\}; \end{aligned} \tag{4.7}$$

(2) if $2 \leqslant k \leqslant \alpha + \frac{1}{p}$, then

$$\mathsf{Alg}\, A_k = \big\{ R:\ c_1 = \dots = c_r \in \mathbb{C}; \quad r_j \in W_{p,0}^{k-1}[0,1], \quad 1 \leqslant j \leqslant r \big\}.$$

Proof. (i) Theorem 2.19 and Corollary 3.2 imply that $\mathbb{O} \oplus \dots \oplus \mathbb{O} \oplus \lambda_j J_{k,0}^\alpha \oplus \mathbb{O} \oplus \dots \oplus \mathbb{O} \in \mathsf{Alg}(\bigoplus_{j=1}^r \lambda_j J_{k,0}^\alpha)$ for any $j \in \{1,\dots,r\}$. It easily implies that $M_j := \mathbb{O} \oplus \dots \oplus \mathbb{O} \oplus\ (\lambda_j J_k^\alpha)^{k+1} \oplus \mathbb{O} \oplus \dots \oplus \mathbb{O} \in A^k\, \mathsf{Alg}\, A_k$. Thus $M_j \in \mathsf{Alg}\, A_k$ and (1.12) implies that if either $\alpha \in \mathbb{Z}_+ \setminus \{0\}$ or $\alpha > k - \frac{1}{p}$, then

$$\mathsf{Alg}\, A_k \supset \big\{ R:\ c_1 = \dots = c_r \in \mathbb{C};\ r_j \in W_{p,0}^{k-1}[0,1],\ \ 1 \leqslant j \leqslant r \big\}. \tag{4.8}$$

(ii) Let $2 \leqslant k \leqslant \alpha + \frac{1}{p}$. Then combining the obvious inclusion $\mathsf{Alg}\, A_k \subset \bigoplus_{j=1}^r \mathsf{Alg}\, \lambda_j J_k^\alpha$ with (1.12) we arrive at opposite inclusion in (4.8). Thus, (2) is proved.

(iii) Let us prove the inclusion "$\subset$" in (4.7). Description (1.12) and inclusion $\mathsf{Alg}\, A_k \subset \bigoplus_{j=1}^r \mathsf{Alg}\, \lambda_j J_k^\alpha$ imply that

$$\begin{aligned} \mathsf{Alg}\, A_k \subset \big\{ R:\ & c_j \in \mathbb{C};\ r_j \in W_p^{k-1}[0,1], \\ & r_j^{(l)}(0) = 0, \quad l \neq \alpha m - 1, \quad m \leqslant [(k-1)/\alpha],\ 1 \leqslant j \leqslant r \big\}. \end{aligned}$$

For $j \in \{1,\dots,r\}$ and $m \in \{1,\dots,[\frac{k-1}{\alpha}]\}$ by definition, put :

$$x_{jm} := \underbrace{0 \oplus \dots \oplus 0 \oplus \mathbf{1}}_{j} \oplus 0 \oplus \dots \oplus 0, \quad y_{jm} := \underbrace{0 \oplus \dots \oplus 0 \oplus \frac{x^{\alpha m}}{\Gamma(\alpha m)}}_{j} \oplus 0 \oplus \dots \oplus 0.$$

Let $R := \bigoplus_{j=1}^r R_j \in \mathsf{Alg}\, A_k$. Choose $\varepsilon_1 > 0$ and put

$$\varepsilon := \min\bigl\{|2^{-1}\lambda_j^m \varepsilon_1| : \ 1 \le j \le r, \ \ 0 \le m \le k_1\bigr\} \qquad \text{and} \qquad k_1 := \Bigl[\frac{k-1}{\alpha}\Bigr].$$

Next, choose vectors $\{x_{jm}\}_{j,m=1}^{r,k_1}$ and $\{y_{jm}\}_{j,m=1}^{r,k_1}$ belonging to $W_p^k[0,1]$ and $(W_p^k[0,1])^* = W_{p'}^k[0,1]$, respectively and define a weak neighborhood $\mathcal{V} := \mathcal{V}(\varepsilon; \{x_{jm}\}_{j,m=1}^{r,k_1}, \{y_{jm}\}_{j,m=1}^{r,k_1})$ of R according to Definition 4.2. Then by definition of $\mathsf{Alg}\, A_k$ there exists a polynomial $p(x) := \sum_{l=0}^N a_l x^l$ such that $p(A_k)$ belongs to the weak neighborhood $\mathcal{V}$ of R, $p(A_k) \in \mathcal{V}$, that is

$$|((R - p(A_k))x_{jm}, y_{jm})| < \varepsilon, \qquad j \in \{1, \dots, r\}, \qquad m \in \{0, \dots, k_1\}. \tag{4.9}$$

It is clear that (4.9) is equivalent to the following system

$$\Bigl|\Bigl(\bigl(R_j - p(\lambda_j J_k^\alpha)\bigr)\mathbf{1}, \frac{x^{\alpha m}}{\Gamma(\alpha m)}\Bigr)\Bigr| < \varepsilon, \quad j \in \{1, \dots, r\}, \quad m \in \{0, \dots, k_1\}.$$

After simple computations this systems reduces to the following one

$$|c_j - a_0| < \varepsilon, \quad \left|\frac{r_j^{(\alpha m-1)}(0)}{\lambda_j^m} - a_m\right| < \frac{\varepsilon}{\lambda_j^m}, \quad j \in \{1, \dots, r\}, \quad m \in \{0, \dots, k_1\}.$$

Finally, triangle inequality implies that

$$|c_1 - c_j| < 2\varepsilon \le \varepsilon_1, \quad \left|\frac{r_1^{(\alpha m-1)}(0)}{\lambda_1^m} - \frac{r_j^{(\alpha m-1)}(0)}{\lambda_i^m}\right| < \frac{2\varepsilon}{\lambda_j^m} \le \varepsilon_1, \quad m \in \{0, \dots, k_1\}.$$

Hence,

$$\begin{aligned}
&c_j = c_1,\\
&r_j^{(\alpha m-1)}(0) = (\lambda_j \lambda_1^{-1})^m r_1^{(\alpha m-1)}(0), \quad m \in \{1, \dots, k_1\}, \quad j \in \{1, \dots, r\}.
\end{aligned}$$

Thus, the inclusion "$\subset$" in (4.7) is proved.

(*iiii*) Let R belongs to the algebra defined by the right side of (4.7). Since $r_j \in W_p^{k-1}[0,1]$, it follows that

$$r_j(x) = r_{j,0} + r_{j,k-2} := \left(r_j(x) - \sum_{i=0}^{k-2} r_j^{(i)}(0)\frac{x^i}{i!}\right) + \sum_{i=0}^{k-2} r_j^{(i)}(0)\frac{x^i}{i!}, \quad j \in \{1, \dots, r\}.$$

According to this decomposition we can write $R = R_0 + R_{k-2}$, where

$$R_0 = R_{1,0} \oplus \dots \oplus R_{r,0}, \qquad R_{k-2} = R_{1,k-2} \oplus \dots \oplus R_{r,k-2},$$

and

$$(R_{j,0}f)(\cdot) := (r_{j,0} * f)(\cdot), \quad (R_{j,k-2}f)(\cdot) := (r_{j,k-2} * f)(\cdot), \qquad j \in \{1, \dots, r\}.$$

Furthermore, $R_0 \in \mathsf{Alg}\, A_k$ by (4.8) and $R_{k-2} \in \mathsf{Alg}\, A_k$ by (*iii*) . Thus (1) is proved. $\square$

Combining Theorems 4.3 and 4.1 we arrive at

Theorem 4.4. *Suppose* $A_k(j) := \bigoplus_{i=1}^{n_j} \lambda_{ji} J_k^\alpha$ *is defined on* $\bigoplus_{i=1}^{n_j} W_p^k[0,1]$, $j \in \{1,\ldots,r\}$ *and* $A_k := \bigoplus_{j=1}^{r} A(j)$ *is defined on* $\bigoplus_{j=1}^{r}(\bigoplus_{i=1}^{n_j} W_p^k[0,1])$. *Let also*

$$\lambda_{ji} = \lambda_{j1}/s_{ji}^\alpha, 1 = s_{j1} \leqslant s_{j2} \leqslant \ldots \leqslant s_{jn_j}, \qquad 1 \leqslant j \leqslant r, \quad 1 \leqslant i \leqslant n_j,$$
$$\arg\lambda_{i1} \neq \arg\lambda_{j1} \pmod{2\pi}, \qquad 1 \leqslant i < j \leqslant r.$$

Let also

$$R := \bigoplus_{j=1}^{r}\bigoplus_{i=1}^{n_j} R_{ji} \in \Big[\bigoplus_{j=1}^{r}\bigoplus_{i=1}^{n_j} W_p^k[0,1]\Big], \quad (R_{ji}f)(\cdot) = (r_{ji} * f)(\cdot), \quad 1 \leqslant j \leqslant r.$$

Then the following are true:

(1) if $1 \leqslant \alpha \leqslant k-1$, *then*

$$\begin{aligned}\mathsf{Alg}\, A_k = \Big\{ & c\mathbb{I} + R: \ c \in \mathbb{C}; \ r_{j1} \in W_p^{k-1}[0,1], \ 1 \leqslant j \leqslant r;\\ & r_{ji}(x) = s_{ji}^{-1} r_{j1}(s_{ji}^{-1}x), \quad 1 \leqslant j \leqslant r, \quad 1 \leqslant i \leqslant n_j;\\ & r_{j1}^{(\alpha m-1)}(0) = (\lambda_{j1}\lambda_{11}^{-1})^m r_{11}^{(\alpha m-1)}(0), \quad m \leqslant \Big[\frac{k-1}{\alpha}\Big], \quad 1 \leqslant j \leqslant r;\\ & r_{j1}^{(l)}(0) = 0, \quad l \neq \alpha m - 1, \ m \leqslant \Big[\frac{k-1}{\alpha}\Big], \quad 1 \leqslant j \leqslant r \Big\};\end{aligned}$$

(2) if $2 \leqslant k \leqslant \alpha + \frac{1}{p}$, *then*

$$\begin{aligned}\mathsf{Alg}\, A_k = \big\{ & c\mathbb{I} + R: \ c \in \mathbb{C}; \ r_{j1} \in W_{p,0}^{k-1}[0,1], \ 1 \leqslant j \leqslant r;\\ & r_{ji}(x) = s_{ji}^{-1} r_{j1}(s_{ji}^{-1}x), \quad 1 \leqslant j \leqslant r, \quad 1 \leqslant i \leqslant n_j \big\}.\end{aligned}$$

Remark 4.5. In this paper we do not consider questions about the reflexivity of the operator A_k. Such results are contained in [17].

4.2 The commutant $\{A_k\}'$

As in Section 2 we define operator $L_a \in [W_p^k[0,1]]$ for $a \in (0,1]$ and $L_a \in [W_{p,0}^k[0,1], W_p^k[0,1]]$ for $a \in (1,\infty)$ by

$$L_a: \ f(x) \to g(x) = \begin{cases} f(ax) & 0 < a \leqslant 1,\\ \begin{cases} 0, & x \in [0, 1-a^{-1}],\\ f(ax - a + 1), & x \in [1-a^{-1}, 1],\end{cases} & a > 1.\end{cases} \tag{4.10}$$

Next we investigate solvability of the equation

$$RJ_k^\alpha = cJ_k^\alpha R \tag{4.11}$$

in the space $X = W_p^k[0,1]$ and describe the set of its solutions. The following proposition plays a crucial role in the sequel. Its proof is based on Corollary 2.21 and use some ideas from [16].

Proposition 4.6. *Let $c \in \mathbb{C}$ and let $R \in [X]$ be a solution of equation (4.11) where $X = W_p^k[0,1]$. Then*

(1) If $c \notin \mathbb{R}_+$, then $R = 0$;

(2) If $0 < c = a^\alpha \leqslant 1$, $a > 0$, then $R \in L_a\{J_k^\alpha\}' = \{J_k^\alpha\}' L_a$, that is,

$$(Rf)(x) = \frac{d}{dx}\int_0^x r(x-t)f(at)\,dt, \qquad r \in W_p^k[0,1];$$

(3) If $1 < c = a^\alpha$, $a > 0$, then $R \in L_a\{J_k^\alpha\}'$, that is,

$$\begin{aligned}(Rf)(x) &= \big(L_a \frac{d}{dx}(r * f)\big)(x)\\ &= \begin{cases} 0, & x \in [0, 1-a^{-1}],\\ a^{-1}\frac{d}{dx}\int\limits_0^{ax-a+1} r(ax-a+1-t)f(t)\,dt, \quad r \in W_{p,0}^k[0,1], & x \in [1-a^{-1}, 1].\end{cases}\end{aligned}$$

Proof. Let $c \in \mathbb{C}$ and $RJ_k^\alpha = cJ_k^\alpha R$. Consider the block matrix representations of the operators J_k^α and R with respect to the direct sum decomposition $W_p^k[0,1] = W_{p,0}^k[0,1] \dotplus X_k$, where $X_k := \operatorname{span}\{1, x, \ldots, x^{k-1}\}$. Since $W_{p,0}^k[0,1] \in \mathsf{Lat}\, J_k^\alpha$, one has

$$J_k^\alpha = \begin{pmatrix} J_{11}^\alpha & J_{12}^\alpha \\ \mathbb{O} & J_{22}^\alpha \end{pmatrix}, \qquad R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}.$$

Now the equality $RJ_k^\alpha = cJ_k^\alpha R$ splits into

$$R_{11}J_{11}^\alpha = cJ_{11}^\alpha R_{11} + cJ_{12}^\alpha R_{21}, \tag{4.12}$$

$$R_{21}J_{11}^\alpha = cJ_{22}^\alpha R_{21}, \tag{4.13}$$

$$\begin{aligned} R_{21}J_{12}^\alpha + R_{22}J_{22}^\alpha &= cJ_{22}^\alpha R_{22},\\ R_{11}J_{12}^\alpha + R_{12}J_{22}^\alpha &= cJ_{11}^\alpha R_{12} + cJ_{12}^\alpha R_{22}.\end{aligned}$$

It is clear that J_{22}^α is a nilpotent operator on X_k and consequently $J_{22}^{\alpha k} = 0$. Therefore one derives from (4.13) that $R_{21}J_{11}^{\alpha k} = cJ_{22}^{\alpha k}R_{21} = \mathbb{O}$. It follows that $R_{21} = \mathbb{O}$ since $\operatorname{ran} J_{11}^{\alpha k}$ is dense in $W_{p,0}^k[0,1]$. Now equation (4.12) takes the form $R_{11}J_{11}^\alpha = cJ_{11}^\alpha R_{11}$, that is, R_{11} intertwines the operators J_{11}^α and cJ_{11}^α.

(1) Let $c \notin \mathbb{R}_+$. Then Corollary 2.21 (i) yields $R_{11} = \mathbb{O}$. Furthermore, since $J^{\alpha k}x^m \in W_{p,0}^k[0,1]$, $m \in \{0, \ldots, k-1\}$, one has

$$0 = R_{11}J_k^{\alpha k}x^m = RJ_k^{\alpha k}x^m = cJ_k^{\alpha k}Rx^m.$$

It follows that $Rx^m = 0$ for $m \in \{0, \ldots, k-1\}$, hence $R = \mathbb{O}$.

(2) Let $0 < c = a^\alpha \leqslant 1$ for some $a > 0$. Then Corollary 2.21 (ii) yields $(R_{11}f)(x) = \frac{d}{dx}\int_0^x r(x-t)f(at)\,dt$, where $r \in L_{p'}[0,1]$. Let us prove that $r \in W_p^k[0,1]$. We have

$$\begin{aligned} a^{\alpha k}(J_k^{\alpha k}R\mathbf{1})(x) &= (RJ_k^{\alpha k}\mathbf{1})(x) = (R_{11}J_k^{\alpha k}\mathbf{1})(x)\\ &= \frac{d}{dx}\int_0^x r(x-t)\frac{(at)^{\alpha k}}{\Gamma(\alpha k+1)}\,dt = a^{\alpha k}(J^{\alpha k}r)(x).\end{aligned}$$

Hence $r = R\mathbf{1} \in W_p^k[0,1]$.

So, the operator R_{11} defined on $W_{p,0}^k[0,1]$ admits a continuation T as an operator defined on $W_p^k[0,1]$ by

$$T:\ W_p^k[0,1] \to W_p^k[0,1], \qquad T:\ f(x) \to \frac{d}{dx}\int_0^x r(x-t)f(at)\,dt.$$

Since $T \restriction W_{p,0}^k[0,1] = R \restriction W_{p,0}^k[0,1] = R_{11}$ and $J_k^{\alpha k}x^m \in W_{p,0}^k[0,1]$ for $m \in \{0,\dots,k-1\}$, we obtain

$$J_k^{\alpha k}Tx^m = a^{-\alpha k}TJ_k^{\alpha k}x^m = a^{-\alpha k}RJ_k^{\alpha k}x^m = J_k^{\alpha k}Rx^m.$$

It follows that $Tx^m = Rx^m$ for $m \in \{0,\dots,k-1\}$. Thus $R = T$.

(3) Since $c = a^\alpha > 1$, Corollary 2.21 (ii) yields

$$\begin{aligned}(R_{11}f)(x) &= \big(L_a \frac{d}{dx}(r*f)\big)(x)\\ &= \begin{cases} 0, & x \in [0, 1-a^{-1}],\\ a^{-1}\frac{d}{dx}\int\limits_0^{ax-a+1} r(ax-a+1-t)f(t)\,dt, & x \in [1-a^{-1},1],\end{cases}\end{aligned}$$

where $r \in L_{p'}[0,1]$. Let us prove that $r \in W_{p,0}^k[0,1]$.

$$\begin{aligned}a^{\alpha k}(J_k^{\alpha k}R\mathbf{1})(x) &= (RJ_k^{\alpha k}\mathbf{1})(x) = (R_{11}J_k^{\alpha k}\mathbf{1})(x)\\ &= \begin{cases} 0, & x \in [0, 1-a^{-1}],\\ a^{-1}\frac{d}{dx}\int\limits_0^{ax-a+1} r(ax-a+1-t)\frac{t^{\alpha k}}{\Gamma(\alpha k+1)}\,dt, & x \in [1-a^{-1},1],\end{cases}\\ &= \begin{cases} 0, & x \in [0, 1-a^{-1}],\\ a^{-1}\frac{d}{dx}(J^{\alpha k+1}r)(ax-a+1), & x \in [1-a^{-1},1],\end{cases}\\ &= \begin{cases} 0, & x \in [0, 1-a^{-1}],\\ (J^{\alpha k}r)(ax-a+1), & x \in [1-a^{-1},1].\end{cases}\end{aligned}$$

Hence

$$(R\mathbf{1})(x) = \begin{cases} 0, & x \in [0, 1-a^{-1}],\\ r(ax-a+1), & x \in [1-a^{-1},1].\end{cases}$$

Since $R\mathbf{1} \in W_p^k[0,1]$, it follows that $r \in W_{p,0}^k[0,1]$.

So, the operator R_{11} defined on $W_{p,0}^k[0,1]$ admits a continuation T on $W_p^k[0,1]$ defined by

$$(Tf)(x) = \begin{cases} 0, & x \in [0, 1-a^{-1}],\\ a^{-1}\frac{d}{dx}\int\limits_0^{ax-a+1} r(ax-a+1-t)f(t)\,dt, & x \in [1-a^{-1},1].\end{cases}$$

Since $T \restriction W_{p,0}^k[0,1] = R \restriction W_{p,0}^k[0,1] = R_{11}$ and $J^{\alpha k}x^m \in W_{p,0}^k[0,1]$ for $m \in \{0,\dots,k-1\}$, one deduces

$$J_k^{\alpha k}Tx^m = a^{-\alpha k}TJ_k^{\alpha k}x^m = a^{-\alpha k}RJ_k^{\alpha k}x^m = J_k^{\alpha k}Rx^m.$$

It follows that $Tx^m = Rx^m$ for $m \in \{0,\dots,k-1\}$. Thus $R = T$. $\square$

Corollary 4.7. *[16, Theorem 3.4]* $R \in \{J_k^\alpha\}'$ *if and only if*

$$(Rf)(x) = \frac{d}{dx}\int_0^x r(x-t)f(t)\,dt = r(0)f(x) + \int_0^x r'(x-t)f(t)\,dt, \quad r \in W_p^k[0,1].$$

Theorem 4.8. *Suppose* $A_k = \bigoplus_{i=1}^n \lambda_i J_k^\alpha$ *is defined on* $X^{(n)} = \bigoplus_{i=1}^n W_p^k[0,1]$ *and*

$$\lambda_i = \lambda_1/s_i^\alpha, \qquad 1 = s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n, \qquad a_{ij} = s_i^{-1}s_j, \qquad 1 \leqslant i,j \leqslant n.$$

Then the commutant $\{A_k\}'$ *is of the form*

$$\{A_k\}' = \{R: \ R = (R_{ij})_{i,j=1}^n, \quad R_{ij} = L_{a_{ij}}K_{ij}\},$$

where

$$(K_{ij}f)(x) = \frac{d}{dx}\int\limits_0^x k_{ij}(x-t)f(t)\,dt, \qquad k_{ij} \in \begin{cases} W_p^k[0,1], & a_{ij} \leqslant 1, \\ W_{p,0}^k[0,1], & a_{ij} > 1. \end{cases}$$

Proof. Let $R = (R_{ij})_{i,j=1}^n$ be the block matrix partition of the operator R with respect to the direct sum decomposition $X^{(n)} = \bigoplus_{i=1}^n W_p^k[0,1]$. Then the equality $RA_k = A_kR$ is equivalent to the following system

$$R_{ij}J_k^\alpha = \lambda_i\lambda_j^{-1}J_k^\alpha R_{ij} = (s_i^{-1}s_j)^\alpha J_k^\alpha R_{ij} = a_{ij}^\alpha J_k^\alpha R_{ij}, \qquad 1 \leqslant i, \ j \leqslant n.$$

To complete the proof it remains to apply Proposition 4.6. $\square$

Theorem 4.9. *Suppose* $A_k = \bigoplus_{j=1}^r \lambda_j J_k^\alpha$ *is defined on* $\bigoplus_{j=1}^r W_p^k[0,1]$ *and* $\arg\lambda_i \neq \arg\lambda_j \pmod{2\pi}$ *for* $1 \leqslant i < j \leqslant r$. *Then the commutant* $\{A_k\}'$ *splits, that is,*

$$\{A_k\}' = \bigoplus_{j=1}^r \{\lambda_j J_k^\alpha\}'.$$

Proof. Following the proof of Theorem 4.8, one arrives at the relations

$$R_{ij}J_k^\alpha = \lambda_i\lambda_j^{-1}J_k^\alpha R_{ij}, \qquad 1 \leqslant i, \ j \leqslant r. \tag{4.14}$$

The latter results with $i = j$ yield $R_{ii} \in \{J_k^\alpha\}'$ for $i \in \{1,\ldots,r\}$, hence by Proposition 4.6 (2)

$$R_{ii}: \ f \to \frac{d}{dx}\int_0^x p_{ii}(x-t)f(t)\,dt, \qquad r_{ii} \in W_p^k[0,1], \qquad i \in \{1,\ldots,r\}.$$

Since $\arg\lambda_i \neq \arg\lambda_j \pmod{2\pi}$ $(1 \leqslant i < j \leqslant r)$, it follows that $\lambda_i\lambda_j^{-1} \notin \mathbb{R}_+$, hence by Proposition 4.6 (1) $R_{ij} = 0$ $(1 \leqslant i \neq j \leqslant r)$. This completes the proof. $\square$

Combining Theorems 4.8 and 4.9, we arrive at

Theorem 4.10. *Suppose* $A_k(j) := \bigoplus_{i=1}^{n_j} \lambda_{ji}J_k^\alpha$ *is defined on* $\bigoplus_{i=1}^{n_j} W_p^k[0,1]$ $j \in \{1,\ldots,r\}$ *and* $A_k := \bigoplus_{j=1}^r A(j)$ *is defined on* $W = \bigoplus_{j=1}^r(\bigoplus_{i=1}^{n_j} W_p^k[0,1])$. *Let also*

$$\begin{aligned} &\arg\lambda_{j1} = \arg\lambda_{ji} \pmod{2\pi}, && 1 \leqslant j \leqslant r, \quad 1 \leqslant i \leqslant n_j, \\ &\arg\lambda_{i1} \neq \arg\lambda_{j1} \pmod{2\pi}, && 1 \leqslant i < j \leqslant r. \end{aligned}$$

Then

$$\{A_k\}' = \bigoplus_{j=1}^r \{A_k(j)\}',$$

where the algebras $\{A_k(j)\}'$ *are described in Theorem 4.8.*

4.3 The double commutant $\{A_k\}''$

Theorem 4.11. *Suppose $A_k = \bigoplus_{i=1}^n \lambda_i J_k^\alpha$ is defined on $W = \bigoplus_{i=1}^n W_p^k[0,1]$ and*

$$\lambda_i = \lambda_1/s_i^\alpha, \quad 1 = s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n, \qquad a_{ij} = s_i^{-1}s_j, \qquad 1 \leqslant i,j \leqslant n.$$

Then

(1)

$$\{A_k\}'' = \big\{c\mathbb{I} + R: \ c \in \mathbb{C}, \ R = \operatorname{diag}(R_1, \ldots, R_n), \ \ (R_i f)(\cdot) = (r_i * f)(\cdot), \\ r_i(x) = s_i^{-1} r_1(s_i^{-1}x), \ \ r_i \in W_p^{k-1}[0,1], \ \ 1 \leqslant i \leqslant n\big\}.$$

(2) The dimension $d_{k,\alpha}$ of the quotient space $\{A_k\}''/\operatorname{\mathsf{Alg}} A_k$ is $d_{k,\alpha} = k-1-[(k-1)/\alpha]$. In particular, $\operatorname{\mathsf{Alg}} A_k = \{A_k\}''$ if and only if either $\alpha = 1$ or $k = 1$.

Proof. Let us set

$$e_i := (\underbrace{0,\ldots,0,1}_{i},0,\ldots,0), \qquad E_{ij} := e_i^T e_j, \qquad 1 \leqslant i,j \leqslant n.$$

Then Theorem 4.8 implies

$$\{A_k\}' = \operatorname{\mathsf{Alg}}\big\{J_k \otimes E_{ii}, \ \ 1 \leqslant i \leqslant n; \\ L_{a_{ij}} \otimes E_{ij}, \ 1 \leqslant j \leqslant i \leqslant n; \ L_{a_{ij}} J_k^k \otimes E_{ij}, \ 1 \leqslant i < j \leqslant n\big\}.$$

Since $\{\bigoplus_{i=1}^n \lambda_i J_k^\alpha\}'' \subset \bigoplus_{i=1}^n \{\lambda_i J_k^\alpha\}''$, it follows from (1.11)

$$\{A_k\}'' \subset \big\{T := (c_1\mathbb{I} + R_1) \oplus \cdots \oplus (c_n\mathbb{I} + R_n): \ c_i \in \mathbb{C}, \\ (R_i f)(\cdot) = (r_i * f)(\cdot), \ \ r_i \in W_p^{k-1}[0,1]\big\}.$$

It is clear that $T(J_k \otimes E_{ii}) = (J_k \otimes E_{ii})T$ for $i \in \{1,\ldots,n\}$. It can easily be checked that

$$T(L_{a_{ij}} \otimes E_{ij}) = (L_{a_{ij}} \otimes E_{ij})T, \qquad 1 \leqslant j \leqslant i \leqslant n, \tag{4.15}$$

$$T(L_{a_{ij}} J_k^k \otimes E_{ij}) = (L_{a_{ij}} J_k^k \otimes E_{ij})T, \qquad 1 \leqslant i < j \leqslant n \tag{4.16}$$

if and only if $c_1 = \cdots = c_n$ and $r_i(x) = s_i^{-1} r_1(s_i^{-1}x)$ for $1 \leqslant i \leqslant n$. Indeed, (4.15) and (4.16) are equivalent to the first and the second of the following relations

$$(c_j\mathbb{I} + R_j)L_{a_{ij}} f = L_{a_{ij}}(c_i\mathbb{I} + R_i)f, \qquad f \in W_p^k[0,1], \qquad 1 \leqslant j \leqslant i \leqslant n,$$

$$(c_j\mathbb{I} + R_j)L_{a_{ij}} J_k^k f = L_{a_{ij}} J_k^k(c_i\mathbb{I} + R_i)f, \qquad f \in W_p^k[0,1], \qquad 1 \leqslant i < j \leqslant n,$$

respectively. According to the definition of $L_{a_{ij}}$ (see (4.10)), we obtain

$$c_j f(a_{ij}x) + \int_0^x r_j(x-t) f(a_{ij}t)\, dt = c_i f(a_{ij}x) + \int_0^{a_{ij}x} r_i(a_{ij}x - t) f(t)\, dt \tag{4.17}$$

for $f \in W_p^k[0,1]$, $x \in [0,1]$ and $1 \leqslant j \leqslant i \leqslant n$, and

$$\begin{aligned}
&c_j(J_k^k f)(a_{ij}x - a_{ij} + 1) + \int\limits_{1-a_{ij}^{-1}}^{x} r_j(x-t)(J_k^k f)(a_{ij}t - a_{ij} + 1)\,dt \\
=&c_i(J_k^k f)(a_{ij}x - a_{ij} + 1) + \int\limits_{0}^{a_{ij}x-a_{ij}+1} r_i(a_{ij}x - a_{ij} + 1)(J_k^k f)(t)\,dt
\end{aligned} \tag{4.18}$$

for $f \in W_p^k[0,1]$, $x \in [1 - a_{ij}^{-1}, 1]$ and $1 \leqslant j < i \leqslant n$.

After simple computations with (4.17)-(4.18), we get

$$\int\limits_0^x \big[r_j(x-t) - a_{ij} r_i(a_{ij}(x-t))\big] f(a_{ij}t)\,dt = (c_i - c_j) f(a_{ij}x),$$

$$\int\limits_0^x \left[r_i(x-t) - a_{ij}^{-1} r_j(a_{ij}^{-1}(x-t))\right] (J_k^k f)(t)\,dt = (c_j - c_i)(J_k^k f)(x).$$

Now it is easy to see that any of the latter equations is equivalent to $c_1 = \cdots = c_n$ and $r_i(x) = s_i^{-1} r_1(s_i^{-1}x)$ for $i \in \{1, \ldots, n\}$. Thus, (1) is proved.

(2) It is clear that $W_p^{k-1}[0,1] \approx W_{p,0}^{k-1}[0,1] \dotplus \operatorname{span}\{\frac{x^l}{l!} : l = 1, \ldots, k-2\}$. Hence (1) implies that

$$\{A_k\}'' \approx \mathbb{C}^1 \dotplus W_p^{k-1}[0,1] \approx \mathbb{C}^1 \dotplus W_{p,0}^{k-1}[0,1] \dotplus \operatorname{span}\Big\{\frac{x^l}{l!} : l = 0, \ldots, k-2\Big\}. \tag{4.19}$$

Further, Theorem 4.1 yields A_k is isomorphic

$$\mathsf{Alg}\, A_k \approx \mathbb{C}^1 \dotplus W_{p,0}^{k-1}[0,1] \dotplus \operatorname{span}\left\{\frac{x^{\alpha m - 1}}{(\alpha m - 1)!} : 1 \leqslant m \leqslant \left[\frac{k-1}{\alpha}\right]\right\}. \tag{4.20}$$

Combining (4.19) with (4.20) we easily arrive at (2). □

Theorem 4.12. *Suppose $A_k = \bigoplus_{j=1}^r \lambda_j J_k^\alpha$ is defined on $\bigoplus_{j=1}^r W_p^k[0,1]$ and $\arg \lambda_i \neq \arg \lambda_j \pmod{2\pi}$ for $1 \leqslant i < j \leqslant r$. Then*

(1) $\{A_k\}'' = \bigoplus_{j=1}^r \{J_k^\alpha\}''$.

(2) The dimension $d_{k,\alpha}$ of the quotient space $\{A_k\}''/\,\mathsf{Alg}\, A_k$ is $d_{k,\alpha} = rk - 1 - [(k-1)/\alpha]$. In particular, $\mathsf{Alg}\, A_k = \{A_k\}''$ if and only if either

(a) $r = 1$ and $\alpha = 1$, or

(b) $r = 1$ and $k = 1$.

Proof. (1) is implied by Theorem 4.9. Furthermore, (1) and Theorem (4.3) imply that

$$\begin{aligned}
\{A_k\}'' &\approx \bigoplus_{j=1}^{r}(\mathbb{C}^1 \oplus W_p^{k-1}[0,1]) \\
&\approx \mathbb{C}^r \ \dot{+} \bigoplus_{j=1}^{r} W_{p,0}^{k-1}[0,1] \dot{+} \bigoplus_{j=1}^{r} \operatorname{span}\Big\{\frac{x^l}{l!} : l = 0, \dots, k-2\Big\} \\
\mathsf{Alg}\, A_k &\approx \mathbb{C}^1 \dot{+} \bigoplus_{j=1}^{r} W_{p,0}^{k-1}[0,1] \dot{+} \operatorname{span}\left\{\frac{x^{\alpha m-1}}{(\alpha m-1)!} : 1 \leqslant m \leqslant \left[\frac{k-1}{\alpha}\right]\right\}.
\end{aligned}$$

Now it is easy to see that $d_{k,\alpha} = r + r(k-1) - 1 - \left[\frac{k-1}{\alpha}\right] = rk - 1 - \left[\frac{k-1}{\alpha}\right]$. Thus (2) is proved. ☐

Combining Theorems 4.11 and 4.12, we obtain

Theorem 4.13. *Under the conditions of Theorem 4.10, we have*

$$\{A_k\}'' = \bigoplus_{j=1}^{r}\{A_k(j)\}'',$$

where the algebras $\{A_k(j)\}''$ are described in Theorem 4.11.

Remark 4.14. Recall that according to celebrated von Neumann theorem $\{T\}'' = \mathsf{Alg}\, T$ whenever T is a normal operator. B. Sz.-Nagy and C. Foias [33]-[34] generalized this result to the wide class of accretive (disssipative) operators. In particular, this result holds for the accretive operator $A = J \otimes B$ defined on $L_2[0,1] \otimes \mathbb{C}^n$, where B is a diagonal positive matrix, $B = B^* > 0$. By Theorem 4.1 this result remains also valid for non-accretive operator $T := A_k = J_k^\alpha \otimes B$ defined on $\bigoplus_{j=1}^n W_2^k[0,1]$, with the same B .

4.4 Invariant subspaces

In [16] we proved that every subspace invariant under J_k^α belongs either to the "continuous chain" $\mathsf{Lat}^c\, J_k^\alpha$ or to the "discrete chain" $\mathsf{Lat}^d\, J_k^\alpha$. It turns out that $\mathsf{Lat}^c\, J_k^\alpha$ does not depend on α: $\mathsf{Lat}^c\, J_k^\alpha = Lat^c J_k$ (see (1.9)). We proved also that the description of $\mathsf{Lat}^d\, J_k^\alpha$ easily follows from that of $\mathsf{Lat}\, J(0,k)^\alpha$. This description is extracted from Theorem 2.1.

In this section we prove that every A_k-invariant subspace can be decomposed into a direct sum of two invariant subspaces : the first one belongs to the "continuous part" of $\mathsf{Lat}\, A_k$ and the second one belongs to the "discrete part" of $\mathsf{Lat}\, A_k$. We show also, that "continuous part" does not depend on α. Moreover, a description of the "discrete part" is deduced from Theorem 2.1.

Let χ_s stand for the characteristic function of an arbitrary nonempty subset $S \subset \mathbb{Z}_n := \{1, \dots, n\}$. We denote by P_S and $\widehat{P_S}$ the canonical projections from

$\bigoplus_{j=1}^n W_p^{k_j}[0,1]$ and from $\bigoplus_{j=1}^n C^{k_j}$ onto $\bigoplus_{j=1}^n \chi_s(j) W_p^{k_j}[0,1]$ and onto $\bigoplus_{j=1}^n \chi_s(j) C^{k_j}$, respectively. Next we let

$$A_{k,S} := \bigoplus_{j=1}^n \chi_s(j)\lambda_j J_{k_j}^\alpha \upharpoonright \operatorname{ran} P_S, \qquad \widehat{A_{k,S}} := \bigoplus_{j=1}^n \chi_s(j)\lambda_j J(0;k_j)^\alpha \upharpoonright \operatorname{ran} \widehat{P_S}$$

and denote by π_S the quotient mapping from $\operatorname{ran} P_S$ onto $\operatorname{ran} \widehat{P_S}$.

Theorem 4.15. *Suppose $A_k = \bigoplus_{j=1}^n \lambda_j J_{k_j}^\alpha$ is defined on $\bigoplus_{j=1}^n W_p^{k_j}[0,1]$ and $\arg\lambda_i \neq \arg\lambda_j \pmod{2\pi}$ for $1 \leqslant i < j \leqslant n$. Then $E \in \mathsf{Lat}\, A_k$ if and only if there exists $S \subset \mathbb{Z}_n$ and $a_1, \ldots, a_n \in [0,1]$ such that*

$$E = \mathsf{Lat}\, A_{k,S} \bigoplus_{j=1}^n \chi_{s^c}(j) E_{a_j,0}^{k_j},$$

where

$$\mathsf{Lat}\, A_{k,S} = \bigcup_M \pi_S^{-1}\left\{[M, (\widehat{A_{k,S}})^{-1}M] \ :\ M \in \mathsf{Lat}\, \widehat{A_{k,S}} \upharpoonright \widehat{A_{k,S}}M\right\} \tag{4.21}$$

and S^c is the complement for S in $\mathbb{Z}_n$ ($S \cup S^c = \mathbb{Z}_n$). Here $[M, (\widehat{A_{k,S}})^{-1}M]$ is a closed interval in the lattice of all subspaces of $\operatorname{ran}\widehat{P_S}$. Each interval satisfies the equation

$$\dim(\widehat{A_{k,S}})^{-1}M - \dim M = \sum_{j\in S} \min\{-[-\alpha], k_j\}. \tag{4.22}$$

Proof. For every $E \in \mathsf{Lat}\, A_k$, we put j in $S := S_E$ if $P_j E \not\subset W_{p,0}^{k_j}[0,1]$ and put j in S^c otherwise. Next we introduce the subspaces $E_S := \operatorname{span}\{A_{k,S}^m P_S E :\ m \geqslant 0\}$ and $E_{S^c} := \operatorname{span}\{A_{k,S^c}^m P_{S^c} E :\ m \geqslant 0\} \subset \bigoplus_{j=1}^n \chi_{s^c}(j) W_{p,0}^{k_j}[0,1]$. It is clear that $E \subset E_S \oplus E_{S^c}$.

Let $M = \max\limits_{1\leqslant j\leqslant n} k_j$. Then the subspace $F := \overline{A_k^M E}$ is invariant for the operator $A_{k,0} := A_k \upharpoonright \bigoplus_{j=1}^n W_{p,0}^{k_j}[0,1]$ and, by Theorem 2.19, $F = \bigoplus_{j=1}^n E_{a_j,0}^{k_j}$ for some $a_j \in [0,1]$. By the construction of S, it is clear that $a_j = 0$ for $j \in S$ and hence

$$F = \left(\bigoplus_{j=1}^n \chi_s(j) W_{p,0}^{k_j}[0,1]\right) \cup \left(\bigoplus_{j=1}^n \chi_{s^c}(j) E_{a_j,0}^{k_j}\right). \tag{4.23}$$

It is clear that $E \supset F \supset E_{S^c}$. Hence $E \supset P_{S^c}E$ and, therefore, $E \supset P_S E$. The latter inclusion yields $E \supset E_S$ and consequently E splits : $E = E_S \oplus E_{S^c}$.

In turn, by Theorem 2.19, E_{S^c} splits: $E_{S^c} = \bigoplus_{j=1}^n \chi_{s^c}(j) E_{a_j,0}^{k_j}$. On the other hand, combining (4.23) with the relations $E = E_S \oplus E_{S^c} \supset F$, one gets $E_S \supset \bigoplus_{j=1}^n \chi_s(j) W_{p,0}^{k_j}[0,1]$. Therefore, $\pi_S(E_S) \in \mathsf{Lat}\, \widehat{A_S}$. Since the quotient map π_S establishes a bijective correspondence between $E_S \in \mathsf{Lat}\, A_S$ with $E_S \supset \bigoplus_{j\in S} W_{p,0}^{k_j}[0,1]$ and $\pi_S(E_S)$, one derives $E_S = \pi_S^{-1}(\pi_S E_S)$. One completes the proof by applying Theorem 2.1. Furthermore, relations (4.21) and (4.22) are implied by the relations (2.1) and (2.2), respectively. $\square$

Corollary 4.16. *[16] Let π be the quotient map*

$$\pi: \ W_p^k[0,1] \to X_k := \ W_p^k[0,1]/W_{p,0}^k[0,1]$$

and $\widehat{J_k^\alpha}$ be the quotient operator on X_k. Then $\mathsf{Lat}\, J_k^\alpha = \mathsf{Lat}^c\, J_k^\alpha \cup \mathsf{Lat}^d\, J_k^\alpha$, where

(a)

$$\mathsf{Lat}^c\, J_k^\alpha = \left\{E_{a,0}^k: \ 0 \leqslant a \leqslant 1\right\}, \ E_{a,0}^k := \left\{f \in W_{p,0}^k[0,1]: \ f(x) = 0, \ x \in [0,a]\right\}$$

is the "continuous part" of $\mathsf{Lat}\, J_k^\alpha$;

(b)

$$\mathsf{Lat}^d\, J_k^\alpha = \pi^{-1}(\mathsf{Lat}\, \widehat{J_k^\alpha}) = \bigcup_M \pi^{-1}\left\{[M, (\widehat{J_k^\alpha})^{-1}M]: \quad M \in \mathsf{Lat}(\widehat{J_k^\alpha} \upharpoonright \widehat{J_k^\alpha}M)\right\}$$

is the "discrete part" of $\mathsf{Lat}\, J_k^\alpha$.

Here $[M, (\widehat{J_k^\alpha})^{-1}M]$ is a closed interval in the lattice of all subspaces of X_k. Each interval satisfies the equation

$$\dim(\widehat{J_k^\alpha})^{-1}M - \dim M = d,$$

where $d = \min\{-[-\alpha], k\}$.

Corollary 4.17. *[16] Operator J_k^α is unicellular if and only if either $\alpha = 1$ or $k = 1$.*

Example. Suppose that the operator $A = \lambda_1 J_{k_1}^\alpha \oplus \lambda_2 J_{k_2}^\alpha$ $(\arg\lambda_1 \neq \arg\lambda_2) \pmod{2\pi}$ is defined on $W_p^{k_1}[0,1] \oplus W_p^{k_2}[0,1]$. By Theorem 4.15, one has the following description of its lattice of invariant subspaces :

$$\begin{aligned}\mathsf{Lat}\, A = &\bigcup_{[a_1,a_2]\in[0,1]\times[0,1]} (E_{a_1,0}^{k_1} \oplus E_{a_2,0}^{k_2}) \cup \bigcup_{a\in[0,1]} \pi_{\{1\}}^{-1}(\mathsf{Lat}\, \widehat{A_{\{1\}}}) \oplus E_{a,0}^k \\ &\cup \bigcup_{a\in[0,1]} E_{a,0}^k \oplus \pi_{\{2\}}^{-1}(\mathsf{Lat}\, \widehat{A_{\{2\}}}) \cup \bigcup \pi_{\{1,2\}}^{-1}(\mathsf{Lat}\, \widehat{A_{\{1,2\}}}),\end{aligned}$$

where lattices $\pi_{\{1\}}^{-1}(\mathsf{Lat}\, A_{\{1\}}) = \mathsf{Lat}^d\, J_{k_1}^\alpha$ and $\pi_{\{2\}}^{-1}(\mathsf{Lat}\, A_{\{2\}}) = \mathsf{Lat}^d\, J_{k_2}^\alpha$ are described in Corollary 4.16. For example, if $k_1 = 1$, $k_2 = 2$, $\lambda_1 = i$, $\lambda_2 = 1$ and $\alpha = 1$, one has $\pi_{\{1\}}^{-1}(\mathsf{Lat}\, \widehat{A_{\{1\}}}) = \mathsf{Lat}^d\, J_1^1 = W_{p,0}^1[0,1] \cup W_p^1[0,1]$, $\pi_{\{2\}}^{-1}(\mathsf{Lat}\, \widehat{A_{\{2\}}}) = \mathsf{Lat}^d\, J_2^1 = W_{p,0}^2[0,1] \cup E_1^2 \cup W_p^2[0,1]$. It is easily seen that $\widehat{A_{\{1,2\}}} = 0 \oplus J(0;2)$, hence, $\widehat{A_{\{1,2\}}} \upharpoonright \operatorname{ran}(\widehat{A_{\{1,2\}}}): \ e_3 \to 0$ (here $\{e_1, e_2, e_3\}$ is the standard basis in $\mathbb{C}^3$). Thus, by Theorem 2.1,

$$\begin{aligned}\mathsf{Lat}\, \widehat{A_{\{1,2\}}} &= \bigcup_{M\subset\{e_3\}} [M, (\widehat{A_{\{1,2\}}})^{-1}M] = [0, \{e_1, e_3\}] \cup [\{e_3\}, \{e_1, e_2, e_3\}] \\ &= \{0\} \cup \bigcup_{\alpha,\beta\in\mathbb{C}} \{\alpha e_1 + \beta e_3\} \cup \bigcup_{\alpha,\beta\in\mathbb{C}} \{\alpha e_1 + \beta e_2, e_3\} \cup \{e_1, e_2, e_3\} \\ &\approx \{0\} \cup \bigcup_{\alpha,\beta\in\mathbb{C}} \{(\alpha, \beta x)\} \cup \bigcup_{\alpha,\beta\in\mathbb{C}} \{(\alpha, \beta), (0, x)\} \cup \{(1,0), (0,1), (0,x)\}.\end{aligned}$$

Hence

$$\begin{aligned}\pi^{-1}_{\{1,2\}}(\operatorname{\mathsf{Lat}}\widehat{A_{\{1,2\}}}) &= \big(W^1_{p,0}[0,1]\oplus W^2_{p,0}[0,1]\big)\\ &\cup \bigcup_{\alpha,\beta\in\mathbb{C}}\big\{\{f_1,f_2\}:\ f_1\in W^1_p[0,1], f_2\in E^2_1, \alpha f_1(0)+\beta f_2'(0)=0\big\}\\ &\cup \bigcup_{\alpha,\beta\in\mathbb{C}}\big\{\{f_1,f_2\}:\ f_1\in W^1_p[0,1], f_2\in W^2_p[0,1], \alpha f_1(0)+\beta f_2(0)=0\big\}\\ &\cup \big(W^1_p[0,1]\oplus W^2_p[0,1]\big).\end{aligned}$$

Remark 4.18. (i) An alternative description of $\mathsf{Lat}^d\, J^\alpha_k$ might be obtained from the Halmos description of $\mathsf{Lat}\, T$ for $T\in[\mathbb{C}^n]$ (see Theorem 2.2).

(ii) A quite different proof of the description of $\mathsf{Lat}\, J_k$ has been originally obtained by E.Tsekanovskii [46].

4.5 Hyperinvariant subspaces

To present a description of $\mathsf{HypLat}\, A_k$ we keep the notation from Subsection 4.4.

Theorem 4.19. *Let the conditions of Theorem 4.8 hold. Then*

$$\mathsf{HypLat}\, A_k = \bigcup_{S\subset\mathbb{Z}_n}\{E_{S^c}\oplus E_S\}.$$

Here

(a) "the continuous part" E_{S^c} is of the form

$$E_{S^c} = \left\{\bigoplus_{j=1}^n \chi_{S^c}(j)E^k_{a_j,0}:\ a=\{a_j\}_{j\in S^c}\in P(\{s_j\}_{j\in S^c})\right\},$$

where

$$P(\{s_i\}_{i\in S^c}) := P(s_{n_1},\dots,s_{n_{|S^c|}}) = \Big\{(a_{n_1},\dots,a_{n_{|S^c|}})\in\square_{|S^c|}:\\ s_{n_j}a_{n_{j+1}}\leqslant s_{n_{j+1}}a_{n_j}\leqslant s_{n_{j+1}}-s_{n_j}+s_{n_j}a_{n_{j+1}},\ 1\leqslant j\leqslant |S^c|-1\Big\}.$$

(b) "the discrete part" E_S is of the form $E_S=\bigoplus_{j=1}^n\chi_S(j)E^k_{l_j}$, where $1\leqslant l_j\leqslant k-1$ and $l_j\leqslant l_i$ if $s_j\leqslant s_i$ for $1\leqslant i,j\leqslant n$;

In particular, if $\lambda_1=\dots=\lambda_n$, then

$$\mathsf{HypLat}\, A_k = \bigcup_{S\subset\mathbb{Z}_n,\ a\in[0,1],\ 1\leqslant l\leqslant k-1}\left\{\bigoplus_{j=1}^n\chi_s(j)E^k_l\bigoplus_{i=j}^n\chi_{s^c}(j)E^k_{a,0}\right\}.$$

Proof. It is clear that $\mathsf{HypLat}\, A_k = \mathsf{HypLat}(\bigoplus_{j=1}^n \lambda_j J_k^\alpha) \subset \bigoplus_{j=1}^n \mathsf{HypLat}\, \lambda_j J_k^\alpha = \bigoplus_{j=1}^n \mathsf{Lat}\, \lambda_j J_k$. Hence if $E \in \mathsf{HypLat}\, A_k$ then $E = \bigoplus_{j=1}^n E_j$, where $E_j \in \mathsf{Lat}\, J_k$. For each $E_j \in \mathsf{Lat}\, J_k$ $(1 \leqslant j \leqslant n)$ we put j in S if $E_j \in \mathsf{Lat}^d\, J_k \backslash W_{p,0}^k[0,1]$ and put j in S^c otherwise (i.e., if $E_j \in \mathsf{Lat}^c\, J_k$). Thus $E = E_S \oplus E_{S^c}$, where $E_{S^c} = \bigoplus_{j=1}^n \chi_S(j) E_{a_j,0}^k$ and $E_S = \bigoplus_{j=1}^n \chi_{S^c}(j) E_{l_j}^k$. Now E_{S^c} is described in Theorem 2.11 and Corollary 3.2. Let us prove that $E_S = \bigoplus_{j=1}^n \chi_{S^c}(j) E_{l_j}^k \in \mathsf{HypLat}\, A_k$ if and only if $l_j \leqslant l_i$ whenever $s_j \leqslant s_i$ for $1 \leqslant i, j \leqslant n$.

Let $s_j \leqslant s_i$ and $P \in \{A_k\}'$ be such that the block matrix partition of the operator P with respect to the direct sum decomposition $\bigoplus_{j=1}^n W_p^k[0,1]$ contains the only non-zero element $P_{ij} := L_{a_{ij}}$. Then the inclusion $PE_S \subset E_S$ yields $E_{l_j} = P_{ij} E_{l_j} \subset E_{l_i}$. So $s_j \leqslant s_i$ yields $E_{l_j} \subset E_{l_i}$ or $l_j \leqslant l_i$.

The opposite statement may be obtained using routine matrix calculations, which we omit. $\square$

Theorem 4.20. *Under the conditions of Theorem 4.9, the lattice* $\mathsf{HypLat}\, A_k$ *splits :*

$$\mathsf{HypLat}\, A_k = \bigoplus_{j=1}^n \mathsf{HypLat}\, \lambda_j J_k^\alpha = \bigoplus_{j=1}^n \mathsf{Lat}\, J_k.$$

Remark 4.21. It is well known (see [11]) that for two bounded operators T_1 and T_2 the splitting of $\mathsf{Lat}(T_1 \oplus T_2)$ implies the splitting of $\mathsf{HypLat}(T_1 \oplus T_2)$. In other words, the relation $\mathsf{Lat}(T_1 \oplus T_2) = \mathsf{Lat}\, T_1 \oplus \mathsf{Lat}\, T_2$ yields the relation $\mathsf{HypLat}(T_1 \oplus T_2) = \mathsf{HypLat}\, T_1 \oplus \mathsf{HypLat}\, T_2$. Theorem 4.20 demonstrates that the converse implication is not true. Nevertheless the converse implication is true for C_0 contractions T_1 and T_2 defined on Hilbert space ([11]).

Summing up Theorems 4.19 and 4.20, we obtain

Theorem 4.22. *Under the conditions of Theorem 4.10, we have*

$$\mathsf{HypLat}\, A_k = \bigoplus_{j=1}^r \mathsf{HypLat}\, A_k(j),$$

where the lattices $\mathsf{HypLat}\, A_k(j)$ *are described in Corollary 4.19.*

4.6 Cyclic subspaces

Some results of this subsection were announced in [13]. First, we present the following simple

Lemma 4.23. *Let* $A \in [\mathbb{C}^k]$, $\sigma(A) = \{0\}$ *and* $P_{\ker A^*}$ *be the orthoprojection from* $\mathbb{C}^k$ *onto* $\ker A^*$. *Then*

(1) $\mu_A = \operatorname{disc} A = \dim(\ker A^*) = \dim(\ker A)$;

(2) $E \in \mathsf{Cyc}\, A$ *if and only if* $PE = \ker A^*$.

Proof. Necessity. Note that $\operatorname{span}\{E, \operatorname{ran} A\} \supset \operatorname{span}\{A^j E : \ j \geqslant 0\}$ and $(\mathbb{I}_k - P_{\ker A^*})E = P_{\operatorname{ran} A} E \subset \operatorname{ran} A$. Therefore, since $E \in \mathsf{Cyc}\, A$, we have

$$\begin{aligned}\mathbb{C}^k &= \operatorname{span}\{A^j E : \ j = 0, 1, \dots, k-1\} \subset \operatorname{span}\{P_{\ker A^*} E, (\mathbb{I}_k - P_{\ker A^*})E, \operatorname{ran} A\} \\ &= \operatorname{span}\{P_{\ker A^*} E, \operatorname{ran} A\} \subset \operatorname{span}\{\ker A^*, \operatorname{ran} A\} = \ker A^* \oplus \operatorname{ran} A = \mathbb{C}^k.\end{aligned}$$

Hence $P_{\ker A^*} E = \ker A^*$.

Sufficiency. Let $PE = \ker A^*$. Then

$$\mathbb{C}^k = \operatorname{span}\{P_{\ker A^*} E, \operatorname{ran} A\} \subset \operatorname{span}\{E, (\mathbb{I}_k - P_{\ker A^*})E, \operatorname{ran} A\} = \operatorname{span}\{E, \operatorname{ran} A\}.$$

Applying the operator A^j, we obtain $\operatorname{ran} A^j = \operatorname{span}\{A^j E, \operatorname{ran} A^{j+1}\}$ $(1 \leqslant j \leqslant k-1)$. Hence

$$\mathbb{C}^k = \operatorname{span}\{E, \operatorname{ran} A\} = \operatorname{span}\{E, AE, \operatorname{ran} A^2\} = \cdots = \operatorname{span}\{E, \dots, A^{k-1} E\}.$$

It means $E \in \mathsf{Cyc}\, A$. ☐

For every system $\phi = \{\overrightarrow{\phi_l}\}_1^N$, $\overrightarrow{\phi_l} \in \mathbb{C}^n$, we denote by $W(\phi)$ the $n \times N$ matrix consisting of the columns $\overrightarrow{\phi_l}$: $W(\phi) = (\overrightarrow{\phi_1}, \dots, \overrightarrow{\phi_N})$.

Corollary 4.24. *Suppose that $A = \bigoplus_{j=1}^n \lambda_j J(0; k_j)^\alpha$ is defined on $\bigoplus_{j=1}^n \mathbb{C}^{k_j}$ and $m_j := \min(-[-\alpha], k_j)$ for $1 \leqslant j \leqslant n$. Then*

(1) $\mu_A = \operatorname{disc} A = \sum_{j=1}^n m_j$;

(2) the following system

$$\overrightarrow{\phi_l} = \operatorname{col}(\phi_{l11}, \dots, \phi_{l1k_1}, \phi_{l21}, \dots, \phi_{l2k_2}, \dots\dots\ \phi_{ln1}, \dots, \phi_{lnk_n}), \qquad 1 \leqslant l \leqslant N$$

generates a cyclic subspace for the operator A

if and only if

(1) $N \geqslant \sum_{j=1}^n m_j$;

(2) the matrix $W_0 = P_{\ker A^*} W(\phi)$ is of maximal rank, that is, $\operatorname{rank} W_0 = \sum_{j=1}^n m_j$.

Theorem 4.25. *Suppose $A_k = \bigoplus_{j=1}^n \lambda_j J_{k_j}^\alpha$ is defined on $\bigoplus_{j=1}^n W_p^{k_j}[0,1]$ and $m_j := \min(-[-\alpha], k_j)$ for $1 \leqslant j \leqslant n$. Then*

(1) $\mu_{A_k} = \operatorname{disc} A_k = \sum_{j=1}^n m_j$;

(2) the system $\{f_l(x)\}_{l=1}^N$ of vectors $f_l(x) = \{f_{l1}(x), \dots, f_{ln}(x)\}$ generates a cyclic subspace for A_k if and only if the following conditions hold

(i) $N \geqslant \sum_{j=1}^n m_j$;

(ii) the matrix

$$W(0) = \begin{pmatrix} f_{11}(0) & f_{21}(0) & \dots & f_{N1}(0) \\ f'_{11}(0) & f'_{21}(0) & \dots & f'_{N1}(0) \\ \vdots & \vdots & & \vdots \\ f_{11}^{(m_1-1)}(0) & f_{21}^{(m_1-1)}(0) & \dots & f_{N1}^{(m_1-1)}(0) \\ \vdots & \vdots & & \vdots \\ f_{1n}(0) & f_{2n}(0) & \dots & f_{Nn}(0) \\ f'_{1n}(0) & f'_{2n}(0) & \dots & f'_{Nn}(0) \\ \vdots & \vdots & & \vdots \\ f_{1n}^{(m_n-1)}(0) & f_{2n}^{(m_n-1)}(0) & \dots & f_{Nn}^{(m_n-1)}(0) \end{pmatrix}$$

is of maximal rank, i.e., $\operatorname{rank} W(0) = \sum_{j=1}^n m_j$.

Proof. It is clear that $E \in \mathsf{Cyc}\, A_k$ implies $\pi E \in \mathsf{Cyc}\, \widehat{A_k}$. To prove the converse assertion we choose a subspace $E \subset \bigoplus_{j=1}^n W_p^{k_j}[0,1]$ such that $\pi E \in \mathsf{Cyc}\, \widehat{A_k}$ and denote by $F := \operatorname{span}\{A^j E : \ j \geqslant 0\}$. Since $\pi F = \bigoplus_{j=1}^n \mathbb{C}^{k_j}$, one gets that $F \supset \bigoplus_{j=1}^n W_{p,0}^{k_j}[0,1]$. Therefore, just in the same way as in Theorem 4.15, we obtain that $F = \pi^{-1}(\pi F) = \pi^{-1}(\bigoplus_{j=1}^n \mathbb{C}^{k_j}) = \bigoplus_{j=1}^n W_p^{k_j}[0,1]$, that is, $E \in \mathsf{Cyc}\, A_k$. To complete the proof it suffices to apply Corollary 4.24. ☐

Remark 4.26. For $\alpha = 1$ and $k_1 = \dots = k_n =: k \geqslant 1$, that is, for the operator $A_k = \bigoplus_{j=1}^n \lambda_j J_k$, Theorem 4.25 has been established in [12] by another method.

We emphasize that the description of the set $\mathsf{Cyc}\, A_{k,0}$ essentially differs from that of $\mathsf{Cyc}\, A_k$. Namely, in contrast to the operator $A_{k,0}$, the description of the set $\mathsf{Cyc}\, A_k$ does not depend on the choice of λ_j.

Summing up, we obtain a description of the cyclic subspaces for the operator $A = \bigoplus_{j=1}^m \lambda_j J_{k_j}^\alpha \oplus \bigoplus_{j=m+1}^n \lambda_j J_{k_j,0}^\alpha$ acting on the mixed space $\bigoplus_{j=1}^m W_p^{k_j}[0,1] \oplus \bigoplus_{j=m+1}^n W_{p,0}^{k_j}[0,1]$.

Theorem 4.27. *Suppose that the operators*

$$A_k(1) := \bigoplus_{j=1}^m \lambda_j J_{k_j}^\alpha, \quad A_{k,0}(1) := \bigoplus_{j=1}^m \lambda_j J_{k_j,0}^\alpha$$

and

$$A_{k,0}(2) := \bigoplus_{j=m+1}^n \lambda_j J_{k_j,0}^\alpha, \quad \textit{and} \quad A := A_k(1) \oplus A_{k,0}(2)$$

are defined on

$$X(1) := \bigoplus_{j=1}^m W_p^{k_j}[0,1], \quad X_0(1) := \bigoplus_{j=1}^m W_{p,0}^{k_j}[0,1]$$

and

$$X_0(2) := \bigoplus_{j=m+1}^n W_{p,0}^{k_j}[0,1], \quad \textit{and} \quad X := X(1) \oplus X_0(2),$$

respectively. Furthermore, let $P(1)$ be the canonical projection from $X = X(1) \oplus X_0(2)$ onto $X(1)$. Then

(1) $\mu_A = \max\{\mu_{A_k(1)}, \mu_{A_{k,0}(1)\oplus A_{k,0}(2)}\}$;

(2) $E \in \mathsf{Cyc}\, A$ *if and only if*

(a) $P(1)E \in \mathsf{Cyc}\, A_k(1)$,

(b) $\overline{A^M E} \in \mathsf{Cyc}(A_{k,0}(1) \oplus A_{k,0}(2))$, *where* $M := \max\limits_{1\leqslant j\leqslant m} k_j$. *Furthermore, the set* $\mathsf{Cyc}(A_{k,0}(1) \oplus A_{k,0}(2))$ *is described in Theorems 2.24 and 2.16 and the set* $\mathsf{Cyc}\, A_k(1)$ *is described in Theorem 4.25.*

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